

# Frobenius Manifolds and Central Invariants for the Drinfeld - Sokolov Bihamiltonian Structures

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## Abstract

The Drinfeld - Sokolov construction associates a hierarchy of bihamiltonian integrable systems with every untwisted affine Lie algebra. We compute the complete set of invariants of the related bihamiltonian structures with respect to the group of Miura type transformations.

## 1 Introduction

The problem of classification of integrable systems of evolutionary PDEs

$$\begin{aligned} u_t^i &= K^i(u; u_x, u_{xx}, \dots), \quad i = 1, \dots, n \\ u &= (u^1, \dots, u^n) \in M^n \end{aligned}$$

was studied by many mathematicians in the last 40 years with the help of various techniques; in such a general setup it remains essentially open, although there are already strong results for many particular subclasses of equations (see, for example, [45, 51, 37, 52, 20] and references therein).

Before starting the classification work one has to adopt a definition of complete integrability. For Hamiltonian PDEs

$$K^i(u; u_x, u_{xx}, \dots) = \{u^i(x), H\}, \quad i = 1, \dots, n$$

with a suitable class of the Poisson brackets  $\{ \ , \ }$  and the Hamiltonians  $H$ , one can define integrability, similarly to the finite dimensional case, by

assuming existence of a complete family of commuting Hamiltonians (we do not explain here the notion of completeness, see e.g. in [20]). More specific is the class of *bihamiltonian* evolutionary PDEs admitting two different Hamiltonian descriptions

$$K^i(u; u_x, u_{xx}, \dots) = \{u^i(x), H_1\}_1 = \{u^i(x), H_2\}_2$$

with respect to a *compatible pair* of Poisson brackets (see below). Under certain genericity assumptions existence of a bihamiltonian representation ensures complete integrability (see details in [20, 17]). Thus, the problem of classification of integrable PDEs reduces to the problem of classification of bihamiltonian structures of a suitable class. Even in this bihamiltonian framework the classification problem is still far from being resolved.

In [20, 39, 17] we proposed a kind of a perturbative approach to the classification problem considering the subclass of bihamiltonian PDEs admitting a (formal) expansion with respect to a small parameter  $\epsilon$

$$\begin{aligned} u_t^i &= A_j^i(u)u_x^j + \epsilon [B_j^i(u)u_{xx}^j + C_{jk}^i(u)u_x^j u_x^k] \\ &\quad + \epsilon^2 [D_j^i(u)u_{xxx}^j + E_{jk}^i(u)u_x^j u_{xx}^k + F_{jkl}^i(u)u_x^j u_x^k u_x^l] + \dots, \quad (1.1) \\ i &= 1, \dots, n \end{aligned}$$

(summation over repeated indices will be assumed). Such systems are to be classified with respect to a certain pronilpotent extension of the group of (local) diffeomorphisms of the manifold  $M^n$  that we called the *group of Miura type transformations* (see Section 2 below). On this way we managed to produce a complete set of invariants of the bihamiltonian structures satisfying certain semisimplicity assumptions. The first part of these invariants is a differential-geometric object defined on the manifold  $M^n$  called *flat pencil of metrics*; it describes the bihamiltonian structure of the *hydrodynamic limit*

$$u_t^i = A_j^i(u)u_x^j \quad (1.2)$$

of the system (1.1). The second part comes from the deformation theory of these bihamiltonian structures of hydrodynamic type; it consists of  $n$  functions of one variable called the *central invariants* of the bihamiltonian structure. The main result of the papers [39, 17] says that the flat pencil of metrics along with the collection of central invariants completely characterizes the equivalence class of a semisimple bihamiltonian structure with respect to the group of local Miura type transformations (for the precise formulation see the Theorem 2.6 below). In particular, the systems of bihamiltonian PDEs with all vanishing central invariants are equivalent to the hydrodynamic limit (1.2).

Apart from this trivial case no general results about *existence* of bihamiltonian structures and integrable hierarchies with a given pair

(flat pencil of metrics, collection of central invariants)

is available. The most studied is the class of the so-called *integrable hierarchies of the topological type* motivated by the needs of the theory of Gromov - Witten invariants. For this class the Poisson pencil comes from a semisimple Frobenius structure on the manifold  $M^n$ ; all the central invariants are constants equal to each other. Some partial existence results for integrable hierarchies of the topological type will appear elsewhere [21]. So, for the moment we have decided to review the list of known examples of bihamiltonian PDEs of the form (1.1) in the framework of our theory of flat pencils and central invariants.

First examples of such analysis have been carried out in [39, 17]. In the present paper we will consider the flat pencils of metrics and the central invariants for the bihamiltonian hierarchies constructed by V. Drinfeld and V. Sokolov in [13].

The Drinfeld - Sokolov's celebrated paper [13] gives a very simple construction, in terms of the Poisson reduction procedure, of a hierarchy of integrable PDEs associated with a Kac - Moody Lie algebra and a choice of a vertex on the extended Dynkin diagram. In this paper we will only consider the most well known version of this construction for which the affine Lie algebra is untwisted and the chosen vertex of the Dynkin diagram is  $c_0$  (the one added to the Dynkin diagram of the associated simple Lie algebra). In this case the hierarchy admits a bihamiltonian structure. The importance of this part of the Drinfeld - Sokolov construction became clear after the discovery, due to V. Fateev and S. Lukyanov [23], of the connection of the *second* Poisson structure for the Drinfeld - Sokolov hierarchy with the semiclassical limit  $W_{cl}(\mathfrak{g})$  of the Zamolodchikov's  $W$ -algebra [53]. Moreover, according to the conjecture of Drinfeld, proved by B. Feigin and E. Frenkel (see in [26, 28]) the classical  $W$ -algebra  $W_{cl}(\mathfrak{g})$  arises naturally on the center of the universal enveloping algebra of the affine algebra  $\hat{\mathfrak{g}}'$  of the Langlands dual Lie algebra  $\mathfrak{g}'$  at the critical level.

In all these theories the *first* Poisson structure of the Drinfeld and Sokolov seems to be something superfluous: in the standard definition the classical  $W$ -algebra is defined just as the second Poisson structure of Drinfeld and Sokolov. However, in the framework of our differential-geometric classification approach a single Poisson bracket has essentially no invariants: after extension to Miura-type transformations with complex coefficients any two local Poisson brackets of our class are equivalent [30]; see also [9, 20].

The main result of this paper is the complete description of the flat pencils of metrics and computation of the central invariants for the Drinfeld - Sokolov bihamiltonian structures for all untwisted affine Lie algebras. We prove that the flat pencils of metrics are obtained from the Frobenius structures on the orbit spaces of the corresponding Weyl groups constructed by one of the authors in [14] via the theory of flat structures of K.Saito *et al.* [49, 48]. The central invariants are proved to be all constants; they are identified with  $\frac{1}{48} \times$  the square lengths, with respect to the normalized invariant bilinear form, of the generators in the Cartan subalgebra. In particular, this proves that the Drinfeld - Sokolov integrable hierarchies for the  $A$ ,  $D$  and  $E$  series are equivalent, in the sense of Definition 2.4, to an integrable hierarchy of the topological type.

The plan of the paper is as follows: we first recall in the next section the definitions of the bihamiltonian structures, the associated flat pencils of metrics and central invariants. In Section 3 and Section 4, we also remind some preliminaries from Poisson geometry and the Drinfeld - Sokolov reduction procedure. In Section 5 we formulate the Main Theorem about invariants of the Drinfeld - Sokolov bihamiltonian structures. The proof of this theorem is given in Section 6 for the  $A_n$  hierarchies, in Section 7 for the  $B_n$ ,  $C_n$ ,  $D_n$  hierarchies, and in Section 8 for the hierarchies associated with the exceptional simple Lie algebras (some relevant formulae are given in the Appendices). In the final section we give some concluding remarks; we also give an example of the Drinfeld - Sokolov equation associated with the *twisted* Kac - Moody Lie algebra of the  $A_2^{(2)}$  type not admitting a bihamiltonian structure.

## 2 Central invariants of semisimple bihamiltonian structures

We study bihamiltonian structures of the following form

$$\begin{aligned} \{u^i(x), u^j(y)\}_a &= \{u^i(x), u^j(y)\}_a^{[0]} + \sum_{k \geq 1} \epsilon^k \{u^i(x), u^j(y)\}_a^{[k]}, \\ \{u^i(x), u^j(y)\}_a^{[k]} &= \sum_{l=0}^{k+1} A_{k,l;a}^{ij}(u; u_x, \dots, u^{(l)}) \delta^{(k-l+1)}(x-y) \end{aligned} \tag{2.1}$$

where  $i, j = 1, \dots, n$ ,  $a = 1, 2$ . Here  $u = (u^1, \dots, u^n) \in M$  for some  $n$ -dimensional manifold  $M$ . The dependent variables  $u^1, \dots, u^n$  will be considered as local coordinates on  $M$ . In this paper the manifold  $M$  will be assumed to be diffeomorphic to a ball.

The coefficients  $A_{k,l;a}^{ij}$  in (2.1) are homogeneous degree  $l$  elements of the graded ring  $\mathcal{B}$  of polynomial functions on the jet bundle of  $M$

$$\mathcal{B} = \varinjlim_k \mathcal{B}_k, \quad \mathcal{B}_k = \mathcal{C}^\infty(M)[u_x, u_{xx}, \dots, u^{(k)}],$$

$$\deg \partial_x^k u^i = k.$$

Antisymmetry and Jacobi identity for both brackets as well as the compatibility condition (see below) are understood as identities for formal power series in  $\epsilon$ .

The leading terms of the Poisson brackets form a bihamiltonian structure of hydrodynamic type. The coefficients of this term will be redenoted as follows

$$\{u^i(x), u^j(y)\}_a^{[0]} = g_a^{ij}(u(x))\delta'(x-y) + \Gamma_{k\ a}^{ij}(u(x))u_x^k\delta(x-y), \quad (2.2)$$

$$a = 1, 2.$$

Here for any  $\lambda \in \mathbb{R}$  the symmetric matrix  $(g_2^{ij}(u) - \lambda g_1^{ij}(u))$  is assumed to be nondegenerate for generic  $u \in M$ . The Poisson bracket  $\{ , \}_a^{[0]}$  is called *the dispersionless limit* of the bracket  $\{ , \}_a$  for every  $a = 1, 2$ .

For every  $a = 1, 2$  the map

$$\mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}[[\epsilon]]$$

given by the formula

$$(P, Q) \mapsto \frac{\delta P}{\delta u^i(x)} \Pi_a^{ij} \frac{\delta Q}{\delta u^j(x)}, \quad (2.3)$$

$$P = P(u; u_x, \dots, u^{(p)}), \quad Q = Q(u; u_x, \dots, u^{(q)}) \in \mathcal{B}$$

$$\Pi_a^{ij} = g_a^{ij}(u)\partial_x + \Gamma_{k\ a}^{ij}(u)u_x^k + \sum_{k \geq 1} \epsilon^k \sum_{l=0}^{k+1} A_{k,l;a}^{ij}(u; u_x, \dots, u^{(l)})\partial_x^{k-l+1}$$

defines a Lie algebra structure on the quotient ring

$$\bar{\mathcal{B}} := \mathcal{B}[[\epsilon]]/\text{Im } \partial_x \quad (2.4)$$

where

$$\partial_x = \sum_k u^{i,k+1} \frac{\partial}{\partial u^{i,k}}, \quad u^{i,k} := \frac{\partial^k u^i}{\partial x^k}.$$

In the formula (2.3) summation over repeated indice  $i, j$  is assumed,

$$\frac{\delta}{\delta u^i(x)} = \frac{\partial}{\partial u^i} - \partial_x \frac{\partial}{\partial u_x^i} + \partial_x^2 \frac{\partial}{\partial u_{xx}^i} - \partial_x^3 \frac{\partial}{\partial u_{xxx}^i} + \dots$$

is the Euler - Lagrange operator. The class of equivalence in the quotient space (2.4) of any element  $P(u; u_x, \dots; \epsilon) \in \mathcal{B}$  will be denoted by

$$\int P(u; u_x, \dots; \epsilon) dx \in \bar{\mathcal{B}}$$

and called a *local functional*. Observe that, if  $P$  and  $Q$  are two homogeneous differential polynomials of the degrees  $p$  and  $q$  respectively then their bracket (2.3) will be a homogeneous element of the ring  $\mathcal{B}[[\epsilon]]$  of formal power series in  $\epsilon$  of the degree  $p + q + 1$  if the degree

$$\deg \epsilon = -1$$

is assigned to the indeterminate  $\epsilon$ . So, for an arbitrary local functional of the degree zero

$$H = \int \sum_{k \geq 0} \epsilon^k P_k(u; u_x, u_{xx}, \dots, u^{(k)}) dx, \quad \deg P_k(u; u_x, u_{xx}, \dots, u^{(k)}) = k$$

the Hamiltonian vector field

$$u_t^i = \{u^i(x), H\} = \Pi^{ij} \frac{\delta P}{\delta u^j(x)}$$

is a system of evolutionary PDEs of the form (1.1) for any of the two Poisson structures  $\Pi^{ij} = \Pi_1^{ij}$  or  $\Pi^{ij} = \Pi_2^{ij}$ .

By the definition of a bihamiltonian structure, any linear combination with constant coefficients of the two Poisson brackets must be again a Poisson bracket on  $\bar{\mathcal{B}}$  (the so-called *compatibility* condition). Due to this property an infinite hierarchy of pairwise commuting systems of PDEs of the form (1.1) can be associated with the bihamiltonian structure (see details in [20]).

In the dispersionless limit  $\epsilon \rightarrow 0$  the equations (1.1) become a system of the first order quasilinear PDEs (1.2). The leading term (2.2) gives a bihamiltonian structure of (1.2).

The bihamiltonian structures (2.1) will be considered up to invertible linear transformations with constant coefficients

$$\begin{aligned} \{ , \}_1 &\mapsto \kappa_{11} \{ , \}_1 + \kappa_{12} \{ , \}_2 \\ \{ , \}_2 &\mapsto \kappa_{21} \{ , \}_1 + \kappa_{22} \{ , \}_2 \\ \kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} &\neq 0. \end{aligned} \tag{2.5}$$

The dependence of the associated integrable hierarchy on the changes (2.5) is nontrivial; it simplifies if one allows only triangular transformations

$$\begin{aligned} \{ , \}_1 &\mapsto \kappa_{11} \{ , \}_1 \\ \{ , \}_2 &\mapsto \kappa_{21} \{ , \}_1 + \kappa_{22} \{ , \}_2 \\ \kappa_{11}\kappa_{22} &\neq 0. \end{aligned} \tag{2.6}$$

**Definition 2.1** *A compatible pair of Poisson brackets (2.1) considered modulo triangular transformations (2.6) is called a Poisson pencil.*

The antisymmetry of the Poisson brackets (2.1) gives a system of linear differential constraints for the coefficients. They can be written in a compact form

$$\Pi_a^{ji} = -(\Pi_a^{ij})^\dagger, \quad a = 1, 2.$$

Here the (formally) adjoint to a scalar differential operator

$$L = \sum_k A_k(x) \partial_x^k$$

is defined by

$$L^\dagger = \sum_k (-\partial_x)^k A_k(x). \quad (2.7)$$

The validity of the Jacobi identity for the pencil of Poisson brackets imposes a system of highly nontrivial nonlinear differential equations for the coefficients. We address the classification problem of bihamiltonian structures (2.1) under an additional assumption of *semisimplicity*.

**Definition 2.2** *A Poisson pencil (2.1) is called semisimple if the roots  $\lambda^1(u), \dots, \lambda^n(u)$  of the characteristic equation  $\det(g_2^{ij}(u) - \lambda g_1^{ij}(u)) = 0$  form a system of local coordinates near a generic point  $u \in M$ . They are called the canonical coordinates of the pencil.*

In the canonical coordinates of a semisimple bihamiltonian structure, the leading terms (2.2) diagonalize [27, 17]:

$$g_1^{ij}(\lambda) = f^i(\lambda) \delta_{ij}, \quad g_2^{ij}(\lambda) = \lambda^i f^i(\lambda) \delta_{ij}, \quad i, j = 1, \dots, n \quad (2.8)$$

for some functions  $f^1(\lambda), \dots, f^n(\lambda)$ ,  $\lambda = (\lambda^1, \dots, \lambda^n) \in M$ .

**Definition 2.3** *A Miura-type transformation is a change of variables of the form*

$$u^i \mapsto \tilde{u}^i(u; u_x, u_{xx}, \dots; \epsilon) = F_0^i(u) + \sum_{k \geq 1} \epsilon^k F_k^i(u; u_x, \dots, u^{(k)}) \quad (2.9)$$

where  $F_k^i \in \mathcal{B}$  with  $\deg F_k^i = k$ , and the map  $u \mapsto F_0^i(u)$  is a diffeomorphism of  $M$ .

All Miura-type transformations form a group  $\mathcal{G}(M)$ . It acts by automorphisms on the graded ring  $\mathcal{B}[[\epsilon]]$ . This action commutes with the action of the operator of total  $x$ -derivative  $\partial_x$ . Therefore the action of the group  $\mathcal{G}(M)$  on the Poisson brackets of the form (2.1) is defined. The explicit formula

$$\tilde{\Pi}_a^{kl} = L_i^k \Pi_a^{ij} L_j^{l\dagger}, \quad a = 1, 2$$

involves the operator of linearization of (2.9)

$$L_j^i = \sum_m \frac{\partial \tilde{u}^i(u; u_x, u_{xx}, \dots; \epsilon)}{\partial u^{j,m}} \partial_x^m$$

and the adjoint operator  $L_j^{i\dagger}$  (see (2.7)).

**Definition 2.4** *Two bihamiltonian structures of the form (2.1) are called equivalent if one can be transformed to another by a combination of a Miura-type transformation and a linear change (2.5).*

At the leading order  $\epsilon = 0$  one obtains the tensor law, with respect to the coordinate change  $u^i \mapsto F_0^i(u)$ , for the  $(2, 0)$  symmetric tensors  $g_a^{ij}(u)$  and the associated (contravariant) Levi-Civita connections  $\Gamma_{k\ a}^{ij}(u)$ .

Recall [30, 9] that the signature of the metric  $g^{ij}(u)$  is the only local (i.e.,  $M = B^n$  = a small ball in  $\mathbb{R}^n$ ) invariant of a *single* Poisson bracket with respect to the group  $\mathcal{G}(B^n)$ . The theory of invariants of bihamiltonian structure is more rich.

In order to avoid inessential complications with signs, let us consider the complex situation assuming the manifold  $M$  to be complex analytic and all coefficients of the Poisson brackets and of the Miura-type transformations to be complex analytic functions in  $u$ . Then the complete set of local invariants of semisimple bihamiltonian structures of the form (2.1) consists of

- flat pencil of metrics on  $M$ ;
- collection of  $n$  functions of one variable called *central invariants*.

Flat pencil of metrics on  $M$  is, roughly speaking, a pair of (contravariant) metrics  $g_1^{ij}(u)$ ,  $g_2^{ij}(u)$  such that, at any point  $u \in M$  their arbitrary linear combination

$$a_1 g_1^{ij}(u) + a_2 g_2^{ij}(u)$$

has zero curvature, and the contravariant Christoffel coefficients for the above metric also have the form of the same linear combination

$$a_1 \Gamma_{k\ 1}^{ij} + a_2 \Gamma_{k\ 2}^{ij}$$



(see details in [16]).

In particular, a flat pencil of metrics arises on an arbitrary Frobenius manifold according to the following construction [14, 15]. Recall that an arbitrary Frobenius manifold is equipped with a flat metric  $\langle \cdot, \cdot \rangle$ , a product of tangent vectors  $(a, b) \mapsto a \cdot b$ , and an Euler vector field  $E$ . We put

$$(\cdot, \cdot)_1 := \langle \cdot, \cdot \rangle \quad (2.10)$$

and define the second metric<sup>1</sup> on the cotangent bundle from the equation

$$(\omega_1, \omega_2)_2 = i_E \omega_1 \cdot \omega_2 \quad (2.11)$$

that must be valid for an arbitrary pair of 1-forms on the Frobenius manifold. In this formula the identification of tangent and cotangent spaces at every point is done by means of the *first* metric  $(\cdot, \cdot)_1$ . By means of this identification one defines the product of 1-forms  $\omega_1 \cdot \omega_2$  via the product of tangent vectors.

Similarly to Definition 2.4 we give

**Definition 2.5** *Two flat pencils are called (locally) equivalent if one can be transformed to another by a combination of a (local) diffeomorphism and a linear change (2.5).*

The differential geometry problem of local classification of flat pencils reduces to an integrable system of differential equations (see [17] and references therein).

One thus arrives at the problem of local classification of semisimple Poisson pencils of the form (2.1), (2.2) with a *given* flat pencil of metrics (i.e., with the given leading term (2.2)). The theory of *central invariants* gives a parametrization of the infinitesimal deformation space of the bihamiltonian structure (2.2). We will not recall here the underlined cohomological theory [39, 17]; we only give the computational formulae for the central invariants.

Denote  $P_a^{ij}(\lambda)$  (resp.  $Q_a^{ij}(\lambda)$ ) the components of the tensor  $A_{1,0;a}^{ij}(u)$  (resp.  $A_{2,0;a}^{ij}(u)$ ) in the canonical coordinates. Here  $i, j = 1, \dots, n$ ,  $a = 1, 2$ . The  $i$ -th ( $i = 1, \dots, n$ ) central invariant of the semisimple bihamiltonian structure (2.1) is defined by

$$c_i(\lambda) = \frac{1}{3(f^i(\lambda))^2} \left[ Q_2^{ii}(\lambda) - \lambda^i Q_1^{ii}(\lambda) + \sum_{k \neq i} \frac{(P_2^{ki}(\lambda) - \lambda^i P_1^{ki}(\lambda))^2}{f^k(\lambda)(\lambda^k - \lambda^i)} \right]. \quad (2.12)$$

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<sup>1</sup>It also appeared in [1] under the guise of the operation of convolution of invariants of reflection groups.

**Theorem 2.6** ([17]) *i) Each function  $c_i(\lambda)$  defined in (2.12) depends only on  $\lambda^i$ ,  $i = 1, \dots, n$ . ii) Two semisimple bihamiltonian structures of the form (2.1) with the same leading terms  $\{ , \}_a^{[0]}$ ,  $a = 1, 2$  are equivalent iff they have the same set of central invariants  $c_i(\lambda^i)$ ,  $i = 1, \dots, n$ .*

Note that linear transformations (2.5) yield fractional linear transformations of the canonical coordinates

$$\lambda^i \mapsto \frac{\kappa_{21} + \lambda^i \kappa_{22}}{\kappa_{11} + \lambda^i \kappa_{12}}, \quad i = 1, \dots, n.$$

The transformation law of central invariants is given by

$$c_i \mapsto \Delta^{-1}(\kappa_{11} + \kappa_{12} \lambda^i) c_i, \quad i = 1, \dots, n \quad (2.13)$$

where  $\Delta = \kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}$ . For the unimodular transformations,  $\Delta = 1$ , the half-differentials

$$\Omega_i = c_i(\lambda^i)(d\lambda^i)^{1/2}$$

remain invariant, while for the simultaneous rescalings

$$\{ , \}_a \mapsto \kappa \{ , \}_a, \quad a = 1, 2$$

one has

$$c_i \mapsto \kappa^{-1} c_i, \quad i = 1, \dots, n.$$

Observe that the central invariants do not change when rescaling only the *first* Poisson bracket without changing the second one. Because of this the central invariants of a Poisson pencil are well defined up to a common constant factor.

### 3 Preliminaries from Poisson geometry

Before explaining the Drinfeld - Sokolov procedure let us first recall the classical construction of the linear Poisson bracket on the dual space  $\mathfrak{g}^*$  to a finite dimensional Lie algebra  $\mathfrak{g}$  (the so-called *Lie - Poisson bracket*). It is uniquely defined by the following requirement: given two linear functions  $a, b$  on  $\mathfrak{g}^*$ ,  $a, b \in \mathfrak{g}$ , their Poisson bracket coincides with the commutator in  $\mathfrak{g}$ :

$$\{a, b\} = [a, b]. \quad (3.1)$$

Choosing a basis in the Lie algebra

$$\mathfrak{g} = \text{span}(e_1, \dots, e_N), \quad [e_i, e_j] = \sum_{k=1}^N c_{ij}^k e_k \quad (3.2)$$

one obtains the Lie - Poisson bracket in the associated dual system of coordinates  $(x_1, \dots, x_N)$  on  $\mathfrak{g}^*$  written in the following form

$$\{x_i, x_j\} = \sum_{k=1}^N c_{ij}^k x_k, \quad i, j = 1, \dots, N. \quad (3.3)$$

The Jacobi identity for the linear Poisson bracket (3.3) is equivalent to the Jacobi identity for the Lie algebra (3.2). The Poisson bivector (3.3) will be denoted

$$\pi_{\mathfrak{g}} \in \Lambda^2 T_x \mathfrak{g}^*. \quad (3.4)$$

Linear Hamiltonians

$$H_a(x) = \langle a, x \rangle, \quad a \in \mathfrak{g}, \quad x \in \mathfrak{g}^*$$

generate the coadjoint action of the Lie group  $G$  associated with  $\mathfrak{g}$ :

$$\dot{x} = \{x, H_a\} \Leftrightarrow \langle b, x(t) \rangle = \langle e^{-t \text{ad}^a} b, x(0) \rangle \quad \text{for any } b \in \mathfrak{g}. \quad (3.5)$$

A simple generalization is given by linear inhomogeneous Poisson bracket

$$\{x_i, x_j\} = \sum_{k=1}^N c_{ij}^k x_k + c_{ij}^0. \quad (3.6)$$

It can be interpreted as the Lie - Poisson bracket on the one-dimensional central extension

$$0 \rightarrow \mathbb{C}k \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$$

of the Lie algebra by means of the 2-cocycle

$$c^0(e_i, e_j) = c_{ij}^0, \quad c^0([a, b], c) + c^0([c, a], b) + c^0([b, c], a) = 0. \quad (3.7)$$

Let us now recall the setting of the Marsden - Weinstein Hamiltonian reduction procedure [44, 43]. Given a Poisson manifold  $\mathcal{M}$ , a family of Hamiltonians

$$H_1(x), \dots, H_N(x) \in \mathcal{C}^\infty(\mathcal{M})$$

forming a  $N$ -dimensional Lie subalgebra  $\mathfrak{g}$  in  $\mathcal{C}^\infty(\mathcal{M})$

$$\{H_i, H_j\} = \sum_{k=1}^N c_{ij}^k H_k(x), \quad c_{ij}^k = \text{const}$$

generates a Poisson action on  $\mathcal{M}$  of the connected and simply connected Lie group  $G$  associated with  $\mathfrak{g}$ , assuming that any nontrivial linear combination

of the generators  $H_1, \dots, H_N$  is not a Casimir of the Poisson bracket on  $\mathcal{M}$ . The vector valued function

$$\mathcal{P}(x) = (H_1(x), \dots, H_N(x)) \in \mathfrak{g}^*$$

is called the *moment map* for the Poisson action. The diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathcal{P}} & \mathfrak{g}^* \\ g \downarrow & & \downarrow \text{Ad}^* g \\ \mathcal{M} & \xrightarrow{\mathcal{P}} & \mathfrak{g}^* \end{array}$$

is commutative for any  $g \in G$ .

Given a Hamiltonian  $H \in \mathcal{C}^\infty(\mathcal{M})$  invariant with respect to the action of the group  $G$

$$\{H, H_i\} = 0, \quad i = 1, \dots, N$$

the goal of the reduction procedure is to reduce the order of the Hamiltonian system

$$\dot{x} = \{x, H\} \tag{3.8}$$

i.e., to find a Poisson manifold  $(\mathcal{M}^{\text{red}}, \{ \cdot, \cdot \}_{\text{red}})$  of a lower dimension and a Hamiltonian  $H_{\text{red}} \in \mathcal{C}^\infty(\mathcal{M}^{\text{red}})$  such that problem of integration of the Hamiltonian system (3.8) is reduced to the one for

$$\dot{y} = \{y, H_{\text{red}}\}_{\text{red}}, \quad y \in \mathcal{M}^{\text{red}}.$$

The construction of the reduced space can be given as follows.

Consider a smooth common level surface of the Hamiltonians

$$\mathcal{M}_h := \{x \in \mathcal{M} \mid H_1(x) = h_1, \dots, H_N(x) = h_N\} = \mathcal{P}^{-1}(h)$$

where

$$h = (h_1, \dots, h_N) \in \mathfrak{g}^*$$

is a regular value of the moment map. Denote  $G_h \subset G$  the stabilizer of  $h$  with respect to the coadjoint action of  $G$  on  $\mathfrak{g}^*$ . The Lie algebra  $\mathfrak{g}_h$  of the stabilizer is the kernel of the map

$$\pi_{\mathfrak{g}} : \mathfrak{g} \simeq T_h^* \mathfrak{g}^* \rightarrow T_h \mathfrak{g}^* \simeq \mathfrak{g}^* \tag{3.9}$$

where  $\pi_{\mathfrak{g}}$  is the Poisson bivector (3.4) on  $\mathfrak{g}^*$ . The group  $G_h$  acts freely on  $\mathcal{M}_h$ . Assume this action to be free also on some neighborhood of  $\mathcal{M}_h \subset \mathcal{M}$  and that the orbit space  $\mathcal{M}_h/G_h$  has a structure of a smooth manifold. Define

$$\mathcal{M}_h^{\text{red}} := \mathcal{M}_h/G_h.$$

We will give a construction of the reduced Poisson bracket on  $\mathcal{M}_h^{\text{red}}$  for the simplest case  $G_h = G$ . In this particular case the Poisson brackets of the generators all vanish on  $\mathcal{M}_h$ :

$$\{H_i, H_j\}|_{\mathcal{M}_h} = 0, \quad i, j = 1, \dots, N.$$

Functions on  $\mathcal{M}_h^{\text{red}}$  can be identified with  $G$ -invariant functions on  $\mathcal{M}_h$ . For any two  $G$ -invariant functions  $\alpha, \beta$  on  $\mathcal{M}_h$  denote  $\hat{\alpha}, \hat{\beta}$  arbitrary extensions of these two functions on a neighborhood of  $\mathcal{M}_h$ .

**Definition 3.1** *Under the above assumptions the Poisson bracket on the reduced space  $\mathcal{M}_h^{\text{red}} = \mathcal{M}_h/G$  defined by the formula*

$$\{\alpha, \beta\}_{\text{red}} := \{\hat{\alpha}, \hat{\beta}\}|_{\mathcal{M}_h} \quad (3.10)$$

*is called the reduced Poisson bracket.*

It is easy to see that the rhs of (3.10) is a  $G$ -invariant function on  $\mathcal{M}_h$ . Moreover, the definition does not depend on the extensions  $\hat{\alpha}, \hat{\beta}$  of the  $G$ -invariant functions  $\alpha, \beta$ .

## 4 The Drinfeld - Sokolov reduction

In this section we will briefly outline the main steps of the Drinfeld - Sokolov reduction for the case of untwisted affine Lie algebras. Proofs of all the statements of this section can be found in [13].

Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$ ,  $G$  the associated connected and simply connected Lie group. Fix an invariant bilinear form  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$  on  $\mathfrak{g}$ . The central extension

$$0 \rightarrow \mathbb{C}k \rightarrow \hat{\mathfrak{g}} \rightarrow L(\mathfrak{g}) \rightarrow 0$$

of the loop algebra  $L(\mathfrak{g}) := C^\infty(S^1, \mathfrak{g})$  is defined as the direct sum of vector spaces  $\hat{\mathfrak{g}} = L(\mathfrak{g}) \oplus \mathbb{C}k$  equipped with the following Lie bracket

$$[q(x) + ak, p(x) + bk] = [q(x), p(x)] + \omega(q, p)k.$$

Here the 2-cocycle  $\omega$  is defined by

$$\omega(q, p) = - \int_{S^1} \langle q(x), p'(x) \rangle_{\mathfrak{g}} dx. \quad (4.1)$$

The integral over the circle is normalized in such a way that

$$\int_{S^1} dx = 1.$$

Let  $\mathcal{M} \subset \hat{\mathfrak{g}}^*$  be the subspace of linear functionals taking value  $\epsilon$  at the central element  $k$ . The space  $\mathcal{M}$  can be naturally identified with the space of first order linear differential operators

$$\mathcal{M} = \left\{ \epsilon \frac{d}{dx} + q(x) \mid q(x) \in L(\mathfrak{g}) \right\} \quad (4.2)$$

in such a way that the coadjoint action of the loop group  $\hat{G} = L(G)$  restricted onto  $\mathcal{M}$  is given by the gauge transformations

$$\begin{aligned} \epsilon \frac{d}{dx} + q(x) &\mapsto \text{Ad}_{g(x)} \left( \epsilon \frac{d}{dx} + q(x) \right) \\ &= \epsilon \frac{d}{dx} + \text{Ad}_{g(x)} q(x) + \Omega_{g(x)} \left( \epsilon \frac{dg}{dx} \right) \end{aligned} \quad (4.3)$$

where the  $\mathfrak{g}$ -valued 1-form  $\Omega_g : T_g G \rightarrow T_e G = \mathfrak{g}$  is defined by

$$\Omega_g(X) = -dR_{g^{-1}}X$$

and

$$R_h : G \rightarrow G$$

is the right shift by  $h \in G$ . It is a Poisson action with respect to the standard linear Lie - Poisson bracket on the dual space  $\hat{\mathfrak{g}}^*$  to the Lie algebra  $\hat{\mathfrak{g}}$ . Recall that the restriction of the Poisson bracket on the subspace  $\mathcal{M} \subset \hat{\mathfrak{g}}^*$  in our realization (4.2) is uniquely determined by the following condition: the Poisson bracket of two linear functionals

$$\begin{aligned} H_{a(x)}[q] &= \int_{S^1} \langle a(x), q(x) \rangle_{\mathfrak{g}} dx, & H_{b(x)}[q] &= \int_{S^1} \langle b(x), q(x) \rangle_{\mathfrak{g}} dx \\ a(x), b(x) &\in L(\mathfrak{g}) \end{aligned} \quad (4.4)$$

coincides with the Lie bracket in  $\hat{\mathfrak{g}}$ :

$$\{H_{a(x)}, H_{b(x)}\} = H_{c(x)} + \omega(a, b), \quad c(x) = [a(x), b(x)]. \quad (4.5)$$

Observe that this functional can be also written in the following elegant form

$$\begin{aligned} \{H_{a(x)}, H_{b(x)}\} &= \frac{1}{\epsilon} \int_{S^1} \langle a(x), [b(x), \epsilon \frac{d}{dx} + q(x)] \rangle_{\mathfrak{g}} dx \\ &= -\frac{1}{\epsilon} \int_{S^1} \langle a(x), \epsilon b_x(x) + \text{ad}_{q(x)} b(x) \rangle dx. \end{aligned} \quad (4.6)$$

In these (and also subsequent) formulae we denote  $H[q]$  the value of a functional  $H$  on the operator

$$\epsilon \frac{d}{dx} + q(x) \in \mathcal{M}$$

for brevity.

It is a standard fact from the theory of Lie - Poisson brackets that the linear Hamiltonians (4.4) generate the coadjoint action (4.3). The Drinfeld - Sokolov construction can be interpreted as the Hamiltonian reduction procedure applied to a certain subgroup of the gauge group  $\hat{G}$ .

**Remark 4.1** *Given a basis  $I^1, \dots, I^N$  in  $\mathfrak{g}$  such that*

$$\begin{aligned} [I^i, I^j] &= c_k^{ij} I^k \\ \langle I^i, I^j \rangle_{\mathfrak{g}} &= g^{ij} \end{aligned}$$

*one obtains a system of coordinates*

$$u^i = \langle I^i, \xi \rangle, \quad \xi \in \mathfrak{g}^*, \quad i = 1, \dots, N \quad (4.7)$$

*on the dual space  $\mathfrak{g}^*$ . The Poisson bracket (4.6) can be written in the form*

$$\{u^i(x), u^j(y)\} = \frac{1}{\epsilon} c_k^{ij} u^k(x) \delta(x - y) - g^{ij} \delta'(x - y). \quad (4.8)$$

*This form of the Poisson bracket is similar to (2.1) but the  $\epsilon$ -expansion begins with terms of order  $\epsilon^{-1}$ . These terms will disappear after the reduction.*

We need some preliminaries from the simple Lie algebras theory in order to develop a suitable infinite dimensional analogue of the above construction of Marsden - Weinstein reduction.

Denote  $n$  the rank of the simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$ . Choose a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  and denote  $X_i, H_i, Y_i$  ( $i = 1, \dots, n$ ) a set of Weyl generators of  $\mathfrak{g}$  associated with  $\mathfrak{h} = \text{span}(H_1, \dots, H_n)$ . The generators of the Cartan subalgebra can be identified with the basis of simple coroots

$$H_i = \alpha_i^\vee \in \mathfrak{h}, \quad i = 1, \dots, n \quad (4.9)$$

associated with a given basis of simple roots  $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ . Recall the commutation relations between the generators:

$$\begin{aligned} [H_i, H_j] &= 0, \\ [H_i, X_j] &= A_{ij} X_j, \quad [H_i, Y_j] = -A_{ij} Y_j, \\ [X_i, Y_j] &= \delta_{ij} H_i. \end{aligned} \quad (4.10)$$

Here  $(A_{ij})$  is the Cartan matrix of the Lie algebra  $\mathfrak{g}$ ,

$$A_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle. \quad (4.11)$$

The full set of defining relations in the simple Lie algebra  $\mathfrak{g}$  is obtained by adding to (4.10) the Serre relations

$$(\text{ad } X_i)^{1-A_{ij}} X_j = 0, \quad (\text{ad } Y_i)^{1-A_{ij}} Y_j = 0, \quad i \neq j.$$

The choice of Cartan subalgebra defines the *principal gradation* on  $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{1-h \leq j \leq h-1} \mathfrak{g}^j \quad (4.12)$$

such that  $X_i \in \mathfrak{g}^1$ ,  $Y_i \in \mathfrak{g}^{-1}$ ,  $H_i \in \mathfrak{g}^0$ . Here  $h$  is the Coxeter number of  $\mathfrak{g}$ .

Let  $\mathfrak{n} = \mathfrak{n}^+$ ,  $\mathfrak{n}^-$  be the nilpotent subalgebras generated by  $\{X_i\}$ ,  $\{Y_i\}$  respectively. Introduce also the Borel subalgebras  $\mathfrak{b} = \mathfrak{b}^+ = \mathfrak{n}^+ \oplus \mathfrak{h}$ ,  $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$ .

Let  $N \subset G$  be the subgroup of the Lie group  $G$  associated with the Lie subalgebra  $\mathfrak{n} \subset \mathfrak{g}$ . We will apply the Hamiltonian reduction procedure to the coadjoint action (4.3) of the loop group  $\hat{N} = L(N)$  on the subspace  $\mathcal{M} \subset \hat{\mathfrak{g}}^*$ .

As we already know the coadjoint action (4.3) of the subgroup  $L(N)$  is generated by the linear Hamiltonians of the form

$$H_{v(x)}[q] = \int_{S^1} \langle v(x), q(x) \rangle_{\mathfrak{g}} dx, \quad v(x) \in \hat{\mathfrak{n}} = L(\mathfrak{n}).$$

Therefore the moment map

$$\mathcal{P} : \mathcal{M} \rightarrow \hat{\mathfrak{n}}^* \quad (4.13)$$

associated with the coadjoint action of  $\hat{N}$  is given by the same formula considered as a linear functional on  $\hat{\mathfrak{n}}$

$$\mathcal{P}(q(x))(v(x)) = \int_{S^1} \langle v(x), q(x) \rangle_{\mathfrak{g}} dx, \quad v(x) \in \hat{\mathfrak{n}}. \quad (4.14)$$

As

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$$

and the orthogonal complement of  $\mathfrak{n}$  w.r.t. the invariant bilinear form  $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$  coincides with  $\mathfrak{b}$ , one can identify the dual space  $\mathfrak{n}^*$  with the quotient

$$\mathfrak{n}^* = \mathfrak{g}/\mathfrak{b} \simeq \mathfrak{n}^-. \quad (4.15)$$



Thus the moment map (4.14) can be identified with the orthogonal projection of the  $\mathfrak{g}$ -valued function  $q(x)$  onto the “lower triangular part”

$$\begin{aligned} \mathcal{P}(q(x)) &= \pi^-(q(x)) \\ \text{where } \pi^- : \mathfrak{g} &\rightarrow \mathfrak{n}^- \text{ is the natural projection.} \end{aligned} \quad (4.16)$$

Let us now choose a particular value of the moment map. Let

$$I = \sum_{i=1}^n Y_i \in \mathfrak{n}^- \quad (4.17)$$

be a *principal nilpotent element* (see [35]) of the Lie algebra  $\mathfrak{g}$ . Denote

$$\mathcal{M}^I := \mathcal{P}^{-1}(I) = I + \hat{\mathfrak{b}} \quad (4.18)$$

the level surface of the moment map considering  $I$  as a constant map  $S^1 \rightarrow \mathfrak{n}^-$ .

From the commutation relations (4.10) it follows that the element  $I \in \mathfrak{n}^*$  is invariant with respect to the coadjoint action of  $\mathfrak{n}$ ,

$$[I, \mathfrak{n}] \subset \mathfrak{b}.$$

Therefore the level surface  $\mathcal{M}^I$  is invariant with respect to the gauge action of the nilpotent group  $N$ . By definition the functionals on the quotient  $\mathcal{M}^I/\hat{N}$  are the gauge invariant functionals on  $\mathcal{M}^I$ . We will now construct a “system of coordinates” on the quotient space.

According to the theory of simple Lie algebras [35], the map

$$\text{ad}_I : \mathfrak{n} \rightarrow \mathfrak{b}$$

is injective. We fix a subspace  $V$  of  $\mathfrak{b}$  such that

$$\mathfrak{b} = V \oplus [I, \mathfrak{n}], \quad (4.19)$$

so  $\dim V = \dim \mathfrak{b} - \dim \mathfrak{n} = n$ .

**Proposition 4.2** *The Hamiltonian action of the loop group  $\hat{N}$  on  $\mathcal{M}^I$  is free, namely, each orbit contains a unique operator of the form*

$$\epsilon \frac{d}{dx} + q^{\text{can}}(x) \quad \text{with } q^{\text{can}}(x) \in V.$$

According to this result of [13] the reduced Poisson manifold can be identified with the space of operators written in the canonical form

$$\mathcal{M}^I/\hat{N} \simeq \left\{ \epsilon \frac{d}{dx} + q^{\text{can}}(x) \mid q^{\text{can}}(x) \in V \right\} \quad (4.20)$$

for the given choice of the subspace  $V \subset \mathfrak{b}$  of the form (4.19). Let us now construct a bihamiltonian structure on this reduced manifold.

Let us first do the following trivial observation: given an element  $\alpha \in \mathfrak{g}$ , the formula

$$\begin{aligned} \{H_{a(x)}, H_{b(x)}\}_\lambda &= \frac{1}{\epsilon} \int_{S^1} \langle a(x), [b(x), \epsilon \frac{d}{dx} + q(x) - \lambda \alpha] \rangle_{\mathfrak{g}} dx \\ &= -\frac{1}{\epsilon} \int_{S^1} \langle a(x), \epsilon b_x(x) + \text{ad}_{q(x)} b(x) \rangle dx + \lambda \frac{1}{\epsilon} \int_{S^1} \langle a(x), \text{ad}_\alpha b(x) \rangle_{\mathfrak{g}} dx \end{aligned} \quad (4.21)$$

(cf. (4.6)) defines a Poisson bracket on  $\mathcal{M}$  for an arbitrary  $\lambda$ . Indeed, the translation

$$q(x) \mapsto q(x) - \lambda \alpha$$

for any  $\lambda$  is a Poisson map for a linear Poisson bracket. We obtain thus a Poisson pencil on  $\mathcal{M}$ .

Let us now choose  $\alpha$  to be a generator of the (one-dimensional) centre of the nilpotent subalgebra,

$$\alpha \in \mathfrak{n}, \quad [\alpha, \mathfrak{n}] = 0. \quad (4.22)$$

The functionals on the reduced space  $\mathcal{M}^I/\hat{N}$  can be realized as functionals on  $\mathcal{M}^I$  invariant with respect to the gauge action of  $\hat{N}$ . Let us call them simply gauge invariant functionals for brevity.

**Proposition 4.3** *Given two gauge invariant functionals  $\phi[q], \psi[q]$  on  $\mathcal{M}^I$ , then for any of their extensions  $\hat{\phi}[q], \hat{\psi}[q]$  to  $\mathcal{M}$ , the functional obtained by restricting the Poisson bracket*

$$\{\hat{\phi}, \hat{\psi}\}_\lambda \quad (4.23)$$

*to  $\mathcal{M}^I$  is again a gauge invariant functional on  $\mathcal{M}^I$ .*

According to this result, the projection of the Poisson pencil from  $\mathcal{M}$  to the reduced space  $\mathcal{M}^I/\hat{N}$  is again a Poisson pencil. In principle this completes the Drinfeld - Sokolov construction, although the explicit realization of the bihamiltonian structure on the reduced space strongly depends on the

choice of the subspace  $V$  in (4.19). Changing the subspace yields a Miura-type transformation of the resulting bihamiltonian structure. The resulting bihamiltonian structure

$$\{ , \}_\lambda = \{ , \}_2 - \lambda \{ , \}_1 \quad (4.24)$$

is called the Drinfeld - Sokolov bihamiltonian structure associated to the simple Lie algebra  $\mathfrak{g}$ . The commuting Hamiltonians of the associated integrable hierarchy can be constructed as (formal) spectral invariants of the differential operator

$$\epsilon \frac{d}{dx} + q^{\text{can}}(x) + I - \lambda \alpha. \quad (4.25)$$

In the subsequent sections, we will recall the explicit representations, following [13], of the reduced space and also of the bihamiltonian structures associated to the simple Lie algebras of type  $A-B-C-D$  in terms of pseudo-differential operators.

At the end of this section we also mention an alternative approach to the Drinfeld - Sokolov reduction, due to P. Casati and M. Pedroni [8]. They start from the bihamiltonian structure

$$\begin{aligned} \{H_{a(x)}, H_{b(x)}\}_1 &= - \int_{S^1} \langle a(x), \text{ad}_\alpha b(x) \rangle_{\mathfrak{g}} dx \\ \{H_{a(x)}, H_{b(x)}\}_2 &= - \int_{S^1} \langle a(x), \epsilon b_x(x) + \text{ad}_{q(x)} b(x) \rangle dx \end{aligned}$$

on  $\mathcal{M}$  with  $\alpha$  chosen as in (4.22). Then they apply the procedure of bihamiltonian reduction, inspired by the more general Marsden - Ratiu reduction algorithm [43] that, in this case, consists of the following main steps:

- find all Casimirs of the first Poisson bracket

$$\{ , H_{b(x)} \}_1 = 0.$$

They have the form  $H_{b(x)}$  with  $[\alpha, b(x)] \equiv 0$ ;

- choose a suitable common level surface  $S$  of the Casimirs;
- consider the Lie subalgebra of those Casimirs that the Hamiltonian flows  $\{ , H_{b(x)} \}_2$  are tangent to  $S$ . The quotient of  $S$  over the action of these flows coincides with the Drinfeld - Sokolov reduced space. The generating functions of the commuting Hamiltonians are the Casimirs of the Poisson pencil

$$\{ , \}_2 - \lambda \{ , \}_1$$

restricted onto the reduced space. In a more recent paper [7] this bihamiltonian approach has been also applied to the  $G_2$  hierarchy.

## 5 Formulation of Main Results

As the first result of the present paper, we will identify the dispersionless limit of the Drinfeld - Sokolov bihamiltonian structures with the canonical bihamiltonian structures defined on the jet spaces of the Frobenius manifolds – the orbit spaces of the Weyl groups. To this end, we need first to establish an isomorphism between the reduced manifolds  $\mathcal{M}^I/\hat{N}$  that underline the Drinfeld - Sokolov bihamiltonian structures and the loop spaces of the orbit spaces of the Weyl groups.

As in section 4, let  $X_i, Y_i, H_i$ ,  $i = 1, \dots, n$  be a set of Weyl generators of the simple Lie algebra  $\mathfrak{g}$ .

We specify the choice of the complement of the subspace  $[I, \mathfrak{n}]$  of  $\mathfrak{b}$  that appears in (4.19) so that

$$V = \bigoplus_{j=0}^{h-1} V_j, \quad (5.1)$$

where the subspaces  $V_j$  satisfy

$$V_j \in \mathfrak{b}_j = \mathfrak{b} \cap \mathfrak{g}^j, \quad \mathfrak{b}_j = V_j \oplus [I, \mathfrak{b}_{j+1}]. \quad (5.2)$$

Note that  $V_j$  is not a null space if and only if  $j$  is one of the exponents

$$1 = m_1 \leq m_2 \leq \dots \leq m_n = h - 1$$

of the simple Lie algebra  $\mathfrak{g}$ . For all simple Lie algebras except the ones of  $D_n$  type with even  $n$  the exponents have multiplicity one, i.e.  $\dim V_{m_i} = 1$  and the exponents are distinct. For the  $D_n$  (with even  $n$ ) case, the exponents  $m_i$  for  $i \neq \frac{n}{2}, \frac{n}{2} + 1$  have multiplicity one,  $m_{\frac{n}{2}} = m_{\frac{n}{2}+1} = n - 1$  and  $\dim V_{n-1} = 2$ .

To choose a system of local coordinates of the reduced manifold  $\mathcal{M}^I/\hat{N}$  of (4.20), we fix a canonical form

$$\epsilon \frac{d}{dx} + q^{\text{can}} + I \in \mathcal{M}^I/\hat{N}$$

of the linear operator  $\epsilon \frac{d}{dx} + q + I$  under the gauge action of  $\hat{N}$  such that

$$q^{\text{can}} = \sum_{i=1}^n u^i \gamma_i \in V. \quad (5.3)$$

Here for the exponent  $m_i$  with multiplicity one,  $\gamma_i$  is a basis of the one-dimensional subspace  $V_{m_i}$ ; for the  $D_n$  case with even  $n$ ,  $\gamma_{\frac{n}{2}}, \gamma_{\frac{n}{2}+1}$  is a basis of the 2-dimensional subspace  $V_{n-1}$ . Then  $u^1, \dots, u^n$  form a coordinate system on the space  $V \subset \mathfrak{n}$ .

**Remark 5.1** *The subspace  $V_{h-1} = \mathfrak{b}_{h-1}$  is determined uniquely since*

$$\mathfrak{b}_j = 0 \quad \text{for } j \geq h.$$

*Recall [6] that  $\mathfrak{b}_{h-1}$  coincides with the (one-dimensional) centre of  $\mathfrak{n}$ . We will choose the basic vector  $\gamma_n \in V_{h-1}$  as follows:*

$$\gamma_n = \alpha \tag{5.4}$$

*where the generator  $\alpha$  of the centre of  $\mathfrak{n}$  has been chosen in (4.22) (see also (4.25)).*

According to the results of Section 4 there exists a gauge transformation reducing the linear operator  $\epsilon \frac{d}{dx} + q + I$  to the canonical form,

$$S^{-1}(x) \left( \epsilon \frac{d}{dx} + q + I \right) S(x) = \epsilon \frac{d}{dx} + q^{\text{can}} + I \tag{5.5}$$

where the function  $S(x)$  takes values in the nilpotent group  $N$ . The canonical form  $q^{\text{can}}$  and the reducing gauge transformation  $S(x)$  are determined uniquely from the following recursion procedure<sup>2</sup>

$$[I, S_{i+1}] - q_i^{\text{can}} = \sum_{j=1}^i S_j q_{i-j}^{\text{can}} - q_i - \sum_{j=1}^i q_{i-j} S_j - \epsilon \frac{d S_i}{dx}, \quad i \geq 0. \tag{5.6}$$

Here we use decomposition

$$S = 1 + S_1 + S_2 + \cdots \in \hat{N} \tag{5.7}$$

induced by the principal gradation of  $\mathfrak{g}$  since the exponential map

$$\mathfrak{n} \rightarrow N$$

is a polynomial isomorphism. As it was proved in [13], the reducing transformation and the canonical form are uniquely determined from the recursion relation. Moreover, they are differential polynomials in  $q$ . In particular, the defined above coordinates  $u^1, \dots, u^n$  of  $q^{\text{can}}$  are certain differential polynomials

$$u^i = u^i(q; q_x, \dots, q^{(h-1)}), \quad i = 1, \dots, n. \tag{5.8}$$

---

<sup>2</sup>Strictly speaking, the form we write (5.5) and the recursion relation (5.6) uses a matrix realization of the Lie algebra. See [13] for the formulation of the recursion procedure independent of the matrix realization.

We will now use these differential polynomials for defining a polynomial isomorphism of affine algebraic varieties

$$\mathfrak{h}/W \rightarrow V. \quad (5.9)$$

where  $W = W_{\mathfrak{g}}$  is the Weyl group of the simple Lie algebra  $\mathfrak{g}$ .

Restricting the differential polynomials  $u^i(q; q_x, q_{xx}, \dots)$  to the Cartan subalgebra

$$q = \xi = \sum_{i=1}^n \xi^i \alpha_i^\vee \in C^\infty(S^1, \mathfrak{h})$$

we obtain differential polynomials

$$u^1(\xi; \xi_x, \xi_{xx}, \dots), \dots, u^n(\xi; \xi_x, \xi_{xx}, \dots). \quad (5.10)$$

Define polynomial functions on  $\mathfrak{h}$  by

$$y^i(\xi) = u^i(\xi; 0, 0, \dots) \in \mathbb{C}[\mathfrak{h}^*]. \quad (5.11)$$

**Lemma 5.2** *The functions  $y^i(\xi)$  are  $W$ -invariant homogeneous polynomials of degree  $m_i+1$ . Moreover, they generate the ring of  $W$ -invariant polynomials  $\mathbb{C}[\mathfrak{h}^*]^W$ .*

*Proof* The restriction

$$F(q; q_x, q_{xx}, \dots) \mapsto F(q; 0, 0, \dots) =: f(q)$$

of any gauge invariant polynomial function on the differential operators of the form

$$\epsilon \frac{d}{dx} + q$$

yields a polynomial function on  $\mathfrak{g}$  invariant wrt adjoint action of the Lie group  $G$ . Further restriction onto the Cartan subalgebra establishes an isomorphism

$$S(\mathfrak{g})^G \rightarrow S(\mathfrak{h})^W = \mathbb{C}[\mathfrak{h}^*]^W$$

of the ring of Ad-invariant polynomial functions on  $\mathfrak{g}$  and the ring of  $W$ -invariant polynomial functions on  $\mathfrak{h}$ , according to Chevalley theorem [6]. Furthermore, the homomorphism

$$S(\mathfrak{g})^G \rightarrow S(\mathfrak{b})^N$$

defined by the formula

$$f \mapsto f(I + q), \quad q \in \mathfrak{b}$$

is an isomorphism (see [36], Theorem 1.3). Finally, according to Theorem 1.2 of [36] the adjoint action of the nilpotent group establishes an isomorphism of affine varieties

$$N \times (I + V) \rightarrow I + \mathfrak{b}.$$

Combining these statements we prove that the polynomials  $y^1(\xi), \dots, y^n(\xi)$  generate the ring  $\mathbb{C}[\mathfrak{h}^*]^W$ .

Now let us prove that  $\deg y^i(\xi) = m_i + 1$ . From the above definition, we know that these functions are determined by the following equation obtained from (5.5) by eliminating  $d/dx$

$$e^{\text{ad}_s}(q + I) = q^{\text{can}} + I, \quad (5.12)$$

where  $s \in \mathfrak{n}, q \in \mathfrak{b}, q^{\text{can}} \in V$  have the decomposition

$$s = \sum_{k=1}^{h-1} s_k, \quad q = \sum_{k=0}^{h-1} q_k, \quad q^{\text{can}} = \sum_{i=1}^{h-1} q_i^{\text{can}} \quad (5.13)$$

with  $s_k, q_k \in \mathfrak{b}_k, q_i^{\text{can}} \in V_i$ . Comparing the degree 0 parts of the left and right hand sides of (5.12), we arrive at

$$\text{ad}_I s_1 = q_0. \quad (5.14)$$

Since the map  $\text{ad}_I : \mathfrak{b}_1 \rightarrow \mathfrak{b}_0$  is an isomorphism, we have a unique  $s_1$  satisfying the above equation. Restricting to  $q_0 = \xi$  we see that  $s_1$  depends linearly on  $\xi$ . Continuing this procedure by comparing the degree 1, degree 2 etc. parts, at the  $i$ -th step we arrive at the equation of the form

$$\text{ad}_I s_i + q_{i-1}^{\text{can}} = F_i \quad (5.15)$$

where  $F_i \in \mathfrak{b}_{i-1}$  is a homogeneous polynomial in  $\xi$  of degree  $i$ . If  $i-1$  is not an exponent, then the above equation has a unique solution with  $q_{i-1}^{\text{can}} = 0$  since the map

$$\text{ad}_I : \mathfrak{b}_i \rightarrow \mathfrak{b}_{i-1} \quad (5.16)$$

is an isomorphism [35]. So  $s_i$  will be a homogeneous polynomial in  $\xi$  of degree  $i$ . In the case when  $i-1 = m_k$  is an exponent the map (5.16) is only injective. So the solution  $s_i \in \mathfrak{b}_i, q_{i-1}^{\text{can}} \in V_{i-1}$  of the above equation (5.15) exists and is determined uniquely. The degree of homogeneous polynomials  $s_i(\xi)$  and  $q_{i-1}^{\text{can}}(\xi)$  is equal to

$$\deg s_i(\xi) = \deg q_{i-1}^{\text{can}}(\xi) = \deg F_i(\xi) = i = m_k + 1.$$

Thus the function  $y^k(\xi)$  (or  $y^k(\xi), y^{k+1}(\xi)$ ) when  $m_k$  has multiplicity one (resp. has multiplicity two) is a homogeneous polynomial of degree  $m_k + 1$ . In this way we prove that  $\deg y^i(\xi) = m_i + 1$  for any  $i = 1, \dots, n$ .  $\square$

We obtained an isomorphism of rings

$$\mathbb{C}[V^*] \rightarrow \mathbb{C}[\mathfrak{h}^*]^W.$$

Dualizing we obtain the isomorphism (5.9) of affine algebraic varieties. This induces the isomorphism

$$\left\{ \begin{array}{l} \text{gauge invariant differential} \\ \text{polynomials } f(q; q_x, q_{xx}, \dots) \\ \text{on the space of differential} \\ \text{operators } \epsilon \frac{d}{dx} + q + I, q(x) \in \mathfrak{b} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{differential polynomials} \\ \text{on the affine algebraic} \\ \text{variety } \mathfrak{h}/W \end{array} \right\} \quad (5.17)$$

Recall [14] that the orbit space  $M_{\mathfrak{g}} = \mathfrak{h}/W$  carries a natural structure of a polynomial Frobenius manifold. According to (5.17) the Hamiltonians of Drinfeld - Sokolov hierarchy can be realized as polynomial functions, considered modulo total  $x$ -derivatives, on the jet space of the Frobenius manifold. We want to compute the Drinfeld - Sokolov bihamiltonian structure in terms of  $M_{\mathfrak{g}}$ .

**Theorem 5.3** *Under the isomorphism (5.17), the Drinfeld - Sokolov bihamiltonian structure associated to an untwisted affine Lie algebra  $\hat{\mathfrak{g}}$  is realized as a bihamiltonian structure on the jet space of  $M_{\mathfrak{g}}$ . Its dispersionless limit coincides with the bihamiltonian structure of hydrodynamic type naturally defined on the jet space of the Frobenius manifold by its flat pencil of metrics.*

*Proof* Let us first remind the construction of the flat pencil of metrics on the orbit space  $M_{\mathfrak{g}}$ . Actually, the construction works uniformly for the orbit space of an arbitrary finite Coxeter group  $W$  (in our case  $W = W_{\mathfrak{g}}$ ). For the chosen basis of simple roots  $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$  denote

$$G_{ab} = \langle \alpha_a^\vee, \alpha_b^\vee \rangle_{\mathfrak{g}}, \quad a, b = 1, \dots, n$$

the Gram matrix of the invariant bilinear form. Let

$$(G^{ab}) = (G_{ab})^{-1} \quad (5.18)$$

be the inverse matrix. It gives a (constant) bilinear form on the cotangent bundle  $T^*\mathfrak{h}$ . The projection of the bilinear form onto the quotient  $\mathfrak{h}/W$  defines a bilinear form on  $T^*M_{\mathfrak{g}}$  non-degenerate outside the locus  $\Delta \subset M_{\mathfrak{g}}$  of



singular orbits (the so-called *discriminant* of the Coxeter group  $W$ ). In order to represent this form in the coordinates let us choose the above constructed system of  $W$ -invariant homogeneous polynomials  $y^1(\xi), \dots, y^n(\xi)$  generating the ring  $\mathbb{C}[\mathfrak{h}^*]^W$ . Here  $\xi = \xi^a \alpha_a \in \mathfrak{h}$ . The polynomial function

$$G^{ab} \frac{\partial y^i(\xi)}{\partial \xi^a} \frac{\partial y^j(\xi)}{\partial \xi^b}$$

is  $W$ -invariant for every  $i, j = 1, \dots, n$  and, thus, is a polynomial in  $y^1, \dots, y^n$ . Denote  $g_2^{ij}(y)$  these polynomials,

$$g_2^{ij}(y(\xi)) = G^{ab} \frac{\partial y^i(\xi)}{\partial \xi^a} \frac{\partial y^j(\xi)}{\partial \xi^b}. \quad (5.19)$$

This gives the Gram matrix of the *second* metric on  $T^*M_{\mathfrak{g}}$  in the coordinates  $y^1, \dots, y^n$ . The associated contravariant Christoffel coefficients are polynomials  $\Gamma_{k2}^{ij}(y)$  defined from the equations

$$\Gamma_{k2}^{ij}(y) dy^k = \frac{\partial y^i}{\partial \xi^a} G^{ab} \frac{\partial^2 y^j}{\partial \xi^b \partial \xi^c} d\xi^c. \quad (5.20)$$

To define the first metric, following [48, 49], let us assume that the invariant polynomial  $y^1(\xi)$  has the maximal degree

$$\deg y^1(\xi) = h.$$

Here  $h$  is the Coxeter number of the Lie algebra  $\mathfrak{g}$ . Put

$$g_1^{ij}(y) := \frac{\partial g_2^{ij}(y)}{\partial y^1}, \quad \Gamma_{k1}^{ij}(y) := \frac{\partial \Gamma_{k2}^{ij}(y)}{\partial y^1}. \quad (5.21)$$

This is the first metric and the associated contravariant Christoffels of the flat pencil of metrics (2.10), (2.11) for the Frobenius structure on  $M_{\mathfrak{g}}$ . The second metric of the pencil depends only on the normalization of the invariant bilinear form. The first metric depends on the choice of the invariant polynomial  $y^1(x)$  of the maximal degree. Changing this polynomial yields a rescaling of the first metric; the Frobenius structure will also be rescaled. This rescaling, however, does not change the central invariants (see the end of Section 2).

Let us also remind the algorithm of [14] of reconstruction of the Frobenius structure on the orbit space<sup>3</sup>. Let  $v^1(\xi), \dots, v^n(\xi)$  be a system of *flat*

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<sup>3</sup>This construction was extended in [19, 22] to the orbit spaces of certain extensions of affine Weyl groups, and in [4] to the orbit spaces of some Jacobi groups. More recently I. Satake [50] extended this construction to the orbit spaces of the reflection groups for elliptic root systems for the so-called case of codimension one.

*generators* of the ring of  $W$ -invariant polynomials in the sense of [48, 49]. Geometrically they give a system of flat coordinates for the first metric:

$$\eta^{ij} := (dv^i, dv^j)_1 = \text{const.}$$

Put

$$g^{ij}(v) := (dv^i, dv^j)_2.$$

Then there exists an element  $F(v)$  of the degree  $2h + 2$  in the ring of  $W$ -invariant polynomials such that

$$\eta^{ik}\eta^{jl}\frac{\partial^2 F(v)}{\partial v^k \partial v^l} = \frac{h}{\deg v^i + \deg v^j - 2} g^{ij}(v). \quad (5.22)$$

The third derivatives

$$c_{ij}^k(v) := \eta^{kl} \frac{\partial^3 F(v)}{\partial v^l \partial v^i \partial v^j}$$

are the structure constants of the multiplication on the tangent space  $T_v M$ .

Define a Poisson bracket for two functionals  $\varphi, \psi$  on  $C^\infty(S^1, \mathfrak{h})$  by the formula

$$\{\varphi, \psi\}[\xi] = \int_{S^1} \left\langle \frac{d}{dx} \text{grad}_{\xi(x)} \varphi, \text{grad}_{\xi(x)} \psi \right\rangle_{\mathfrak{g}} dx \quad (5.23)$$

In terms of the coordinates  $\xi^1(x), \dots, \xi^n(x)$ , we have

$$\{\xi^i(x), \xi^n(y)\} = -G^{ij} \delta'(x - y), \quad i, j = 1, \dots, n, \quad (5.24)$$

where  $(G^{ij})$  is defined in (5.18). Then as it is shown in [13], the Miura map

$$\mu : (\xi^1, \dots, \xi^n) \mapsto (u^1(\xi; \xi_x, \xi_{xx}, \dots), \dots, u^n(\xi; \xi_x, \xi_{xx}, \dots))$$

is a Poisson map between  $C^\infty(S^1, \mathfrak{h})$  and  $\mathcal{M}^I / \hat{N}$  if the latter is endowed with the second Poisson bracket of the Drinfeld - Sokolov bihamiltonian structure (4.24).

From the above argument and (5.24), we see that the second metric (5.19) defined on the orbit space of  $W_g$  coincides, up to a minus sign, with the metric defined on  $(I + \mathfrak{b})/N$  by the leading terms of the second Poisson bracket of the Drinfeld - Sokolov bihamiltonian structure associated with the untwisted affine Lie algebra  $\hat{\mathfrak{g}}$ .

The definition of the first Drinfeld - Sokolov Poisson bracket depends on the choice of the base element  $\alpha$  of the one-dimensional center of the nilpotent subalgebra  $\mathfrak{n}$  of  $\mathfrak{g}$ , see (4.21), (4.22). We note that  $\mathfrak{g}^{m_n} = \mathfrak{g}^{h-1}$  is just the center of  $\mathfrak{n}$ , so we can take  $\gamma_n = \alpha$  in (5.3). Then in terms of the

local coordinates  $u^1(x), \dots, u^n(x)$  the first Drinfeld - Sokolov Poisson bracket is obtained from the second one by the shifting

$$u^n(x) \mapsto u^n(x) - \lambda, \quad \partial_x^k u^n(x) \mapsto \partial_x^k u^n(x), \quad k \geq 1,$$

and

$$\{ \cdot, \cdot \}_2 \mapsto \{ \cdot, \cdot \}_2 - \lambda \{ \cdot, \cdot \}_1.$$

Thus from the above results it follows the validity of Theorem 5.5.  $\square$

**Remark 5.4** *Relationship of the generalized Drinfeld - Sokolov hierarchies with algebraic Frobenius manifolds is currently under investigation; first results have been obtained in [47, 11].*

**Theorem 5.5** *The suitably ordered central invariants of the Drinfeld - Sokolov bihamiltonian structure for an untwisted affine Lie algebra  $\hat{\mathfrak{g}}$  are given by the formula*

$$c_i = \frac{1}{48} \langle \alpha_i^\vee, \alpha_i^\vee \rangle_{\mathfrak{g}}, \quad i = 1, \dots, n, \quad (5.25)$$

where  $\alpha_i^\vee \in \mathfrak{h}$  are the coroots of the simple Lie algebra  $\mathfrak{g}$ .

In the formula (5.25) we use the same invariant bilinear form as the one used in the definition of the Kac - Moody Lie algebra in Section 4 .

Let us now comment the statement of Theorem 5.5 regarding the central invariants of the Drinfeld - Sokolov bihamiltonian structures. Let us fix on  $\mathfrak{g}$  the so-called *normalized* invariant bilinear form (see [33], §6.2 and Exercise 6.2)

$$\langle a, b \rangle_{\mathfrak{g}} := \frac{1}{2h^\vee} \text{tr}(\text{ad } a \cdot \text{ad } b). \quad (5.26)$$

Here  $h^\vee$  is the dual Coxeter number. With the help of the table in § 6.7 of [33] one obtains the following values of central invariants, according to Theorem

5.5:

$\mathfrak{g}$	$c_1$	$\dots$	$c_{n-1}$	$c_n$
$A_n$	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{24}$
$B_n$	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{12}$
$C_n$	$\frac{1}{12}$	$\dots$	$\frac{1}{12}$	$\frac{1}{24}$
$D_n$	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{24}$
$E_n, n = 6, 7, 8$	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{24}$
$F_n, n = 4$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{12}$
$G_n, n = 2$	$\frac{1}{8}$			$\frac{1}{24}$

The “breaking of symmetry” between the central invariants for the non-simply laced Lie algebras has the following “experimental” explanation. Recall that the central invariants (2.12) are in one-to-one correspondence with the canonical coordinates on the Frobenius manifold, i.e., with the roots  $\lambda_1, \dots, \lambda_n$  of the characteristic equation

$$\det(g_2^{ij}(u) - \lambda g_1^{ij}(u)) = 0. \quad (5.27)$$

It turns out that the characteristic polynomial factorizes in the product of two factors of the degrees  $p$  and  $q$ ,  $p + q = n$ , where  $p$  is the number of long simple roots and  $q$  is the number of short simple roots. Such a splitting defines a partition of the set of central invariants in two subsets; the central invariants inside each of the subsets have the same value. For simply laced root systems the characteristic polynomial is irreducible. Recall that the map associating with the point  $u$  the collection of the coefficients of the characteristic polynomial (5.27) for the case of simply laced root systems coincides with the *Lyashko - Looijenga map* [41, 42], see also [31].

The values of the central invariants associated to non-simply laced Lie

algebras can be obtained by means of the following “folding prescriptions”:

$$\begin{aligned}
B_n &: \left( \frac{1}{24}, \dots, \frac{1}{24}, \frac{1}{12} \right) = \left( \frac{1}{24}, \dots, \frac{1}{24}, \frac{1}{24} + \frac{1}{24} \right), \\
C_n &: \left( \frac{1}{12}, \dots, \frac{1}{12}, \frac{1}{24} \right) = \left( \frac{1}{24} + \frac{1}{24}, \dots, \frac{1}{24} + \frac{1}{24}, \frac{1}{24} \right), \\
F_4 &: \left( \frac{1}{12}, \frac{1}{12}, \frac{1}{24}, \frac{1}{24} \right) = \left( \frac{1}{24} + \frac{1}{24}, \frac{1}{24} + \frac{1}{24}, \frac{1}{24}, \frac{1}{24} \right), \\
G_2 &: \left( \frac{1}{8}, \frac{1}{24} \right) = \left( \frac{1}{24} + \frac{1}{12}, \frac{1}{24} \right) = \left( \frac{1}{24} + \frac{1}{24} + \frac{1}{24}, \frac{1}{24} \right).
\end{aligned}$$

These prescriptions correspond to the “folding of Dynkin diagrams” procedure known in the theory of simple Lie algebras and singularity theory

$$D_{n+1} \rightarrow B_n, \quad A_{2n+1} \rightarrow C_n, \quad E_6 \rightarrow F_4, \quad D_4 \rightarrow B_3 \rightarrow G_2.$$

Note that the relationships between the Frobenius manifolds associated with simply laced and non-simply laced Coxeter groups established by the folding procedure has been clarified in [54].

Let us take  $B_3 \rightarrow G_2$  as an example to illustrate this relation. Their Dynkin diagrams are



The folding relation means that the simple Lie algebra of type  $B_3$  contains a subalgebra  $\tilde{\mathfrak{g}}$  of type  $G_2$ .

Let  $X_i, \alpha_i^\vee, Y_i$  ( $i = 1, 2, 3$ ) be the Weyl generators for the simple Lie algebra of type  $B_3$  corresponding to the above Dynkin diagram, then the Lie subalgebra generated by

$$\tilde{X}_1 = X_1 + X_3, \tilde{\alpha}_1^\vee = H_1 + H_3, \tilde{Y}_1 = Y_1 + Y_3, \tilde{X}_2 = X_2, \tilde{\alpha}_2^\vee = H_2, \tilde{Y}_2 = Y_2$$

is a simple Lie algebra of type  $G_2$ , and  $\tilde{X}_i, \tilde{\alpha}_i^\vee, \tilde{Y}_i$  ( $i = 1, 2$ ) form a set of Weyl generators of this subalgebra.

Note that in this case these two Lie algebras have the same normalized invariant bilinear forms, so we can compute the central invariants of associated to the simple Lie algebra of type  $G_2$  from that of type  $B_3$  as follows:

$$\begin{aligned}
\tilde{c}_1 &= \frac{\langle \tilde{\alpha}_1^\vee, \tilde{\alpha}_1^\vee \rangle_{\tilde{\mathfrak{g}}}}{48} = \frac{\langle \alpha_1^\vee + \alpha_3^\vee, \alpha_1^\vee + \alpha_3^\vee \rangle_{\mathfrak{g}}}{48} \\
&= \frac{\langle \alpha_1^\vee, \alpha_1^\vee \rangle_{\mathfrak{g}}}{48} + \frac{\langle \alpha_3^\vee, \alpha_3^\vee \rangle_{\mathfrak{g}}}{48} = c_1 + c_3, \\
\tilde{c}_2 &= \frac{\langle \tilde{\alpha}_2^\vee, \tilde{\alpha}_2^\vee \rangle_{\tilde{\mathfrak{g}}}}{48} = \frac{\langle \alpha_2^\vee, \alpha_2^\vee \rangle_{\mathfrak{g}}}{48} = c_2.
\end{aligned}$$

Here we used the fact  $\langle \alpha_1^\vee, \alpha_3^\vee \rangle_{\mathfrak{g}} = 0$ .

In general, when we fold two vertices  $i, j$  in a Dynkin diagram, they must be non-connected. So we have  $\langle \alpha_i^\vee, \alpha_j^\vee \rangle_{\mathfrak{g}} = 0$ . Then the central invariant corresponding to the folded vertex reads

$$\tilde{c} = \frac{\langle \tilde{\alpha}_i^\vee, \tilde{\alpha}_j^\vee \rangle_{\tilde{\mathfrak{g}}}}{48} = \frac{\langle \alpha_i^\vee, \alpha_i^\vee \rangle_{\mathfrak{g}}}{48} + \frac{\langle \alpha_j^\vee, \alpha_j^\vee \rangle_{\mathfrak{g}}}{48} = c_i + c_j.$$

On the other hand, the folding of Dynkin diagrams also establishes relations of the bihamiltonian structures associated to the relevant simple Lie algebras through Dirac reductions. The above mentioned relation between the central invariants and the folding of the Dynkin diagrams provide clues to understand connections of the central invariants of two bihamiltonian structures, with one bihamiltonian structure obtained from the other by Dirac reduction. We will study this aspect in detail in subsequent papers.

The proof of the theorem 5.5 will be given in Section 6 for the  $A_n$  series, in Section 7 for the  $B_n$ ,  $C_n$ ,  $D_n$  series and in Section 8 for the exceptional cases.

## 6 The $A_n$ case

We first recall the Drinfeld - Sokolov bihamiltonian structure related to the simple Lie algebra  $\mathfrak{g}$  of  $A_n$  type. This Lie algebra has the matrix realization  $sl(n+1, \mathbb{C})$ . We denote by  $e_{ij}$  the matrix with 1 at the  $(i, j)$ -th entry and 0 elsewhere. The Weyl generators of  $\mathfrak{g}$  are chosen as

$$X_i = e_{i,i+1}, \quad Y_i = e_{i+1,i}, \quad H_i = e_{i,i} - e_{i+1,i+1}, \quad i = 1, \dots, n. \quad (6.1)$$

We use here the invariant bilinear form

$$\langle a, b \rangle_{\mathfrak{g}} = \text{tr}(ab), \quad (6.2)$$

which coincides with the normalized invariant bilinear form (5.26) on  $\mathfrak{g}$ . The nilpotent subalgebra  $\mathfrak{n}$ , the Borel subalgebra  $\mathfrak{b}$  and the group  $N$  are realized as

$$\begin{aligned} \mathfrak{n} &= \{(a_{ij}) \in \text{Mat}(n+1, \mathbb{C}) \mid a_{ij} = 0, \text{ for } i \geq j\}, \\ \mathfrak{b} &= \{(a_{ij}) \in \text{Mat}(n+1, \mathbb{C}) \mid a_{ij} = 0, \text{ for } i > j\}, \\ N &= \{(s_{ij}) \in \text{Mat}(n+1, \mathbb{C}) \mid s_{ij} = 0 \text{ for } i > j, \text{ } s_{ii} = 1\}. \end{aligned}$$

The element  $I \in \mathfrak{g}$  that is introduced in (4.17) now has the expression  $\sum_{i=1}^n e_{i+1,i}$ . We choose the base element  $\alpha \in \mathfrak{g}$  of the center of  $\mathfrak{n}$ , see (4.22), as

$$\alpha = -e_{1,n+1} \in \mathfrak{n}.$$

Let  $q$  be an element in  $\hat{\mathfrak{b}}$ ,

$$q = \sum_{i=1}^n \sum_{j=i}^{n+1} q_{ij}(x) e_{ij} - \sum_{i=1}^n q_{ii}(x) e_{n+1, n+1}.$$

We can choose the coordinate  $q^{\text{can}}$  on the orbit space (4.20) as [13]

$$q^{\text{can}} = -(u_1(x)e_{1, n+1} + u_2(x)e_{2, n+1} + \cdots + u_n(x)e_{n, n+1}),$$

where  $u_k(x)$  are certain differential polynomials of  $q_{ij}$ . Here and henceforth we use lower indices for the variable  $u$  instead of upper ones as in (5.3) for the convenience of presentation of relevant formulae. Then the gauge invariant functionals take the following form

$$F = \int_{S^1} f(x, u(x), u_x(x), \cdots) dx. \quad (6.3)$$

The space of the gauge invariant functionals can be described in the following way [13]. Consider the operator

$$\mathcal{L} = \epsilon \frac{d}{dx} + q + I \quad (6.4)$$

as a  $(n+1) \times (n+1)$  matrix with entries of differential operators. Let us represent it in the form

$$\mathcal{L} = \begin{pmatrix} \alpha & \beta \\ A & \gamma \end{pmatrix}. \quad (6.5)$$

Here  $A$  is a  $n \times n$  matrix. We can associate to it a scalar differential operator

$$\Delta(\mathcal{L}) := \beta - \alpha A^{-1} \gamma. \quad (6.6)$$

Define

$$L = -\Delta(\mathcal{L})^\dagger, \quad (6.7)$$

where the conjugation of a differential operator is defined as in (2.7). It can be written in the form

$$L = D^{n+1} + u_n(x)D^{n-1} + \cdots + u_2(x)D + u_1(x), \quad D = \epsilon \frac{d}{dx}. \quad (6.8)$$

Gauge invariant functionals on  $\mathcal{M}$  will be identified with functionals on the space of Lax operators (6.8). The variational derivative of a gauge invariant functional  $F$  w.r.t  $L$  is defined as the following pseudo-differential operator

$$\frac{\delta F}{\delta L} = \sum_{i=1}^n D^{-i} \frac{\delta F}{\delta u_i}.$$

It is easy to verify the following identity

$$\delta F = \int \sum_{i=1}^n \frac{\delta F}{\delta u_i(x)} \delta u_i dx = \text{Tr} \left( \frac{\delta F}{\delta L} \delta L \right) \quad (6.9)$$

where the linear functional  $\text{Tr}$  on pseudo-differential operators is defined by

$$\text{Tr} A = \int \text{res} A dx \in \bar{\mathcal{B}}$$

and the residue of a pseudo-differential operator has the definition

$$\text{res} \left( \sum_{i \leq m} f_i D^i \right) = f_{-1}.$$

Recall that, due to the important property of the residue

$$\text{res}(BA) = \text{res}(AB) + \text{total } x\text{-derivative}, \quad (6.10)$$

the formula

$$\text{Tr}(AB) = \int_{S^1} \text{res}(AB) dx \in \bar{\mathcal{B}}$$

defines an invariant symmetric inner product between two pseudo-differential operators.

In terms of the gauge invariant functionals  $F, G$ , the Drinfeld - Sokolov bihamiltonian structure can be written as

$$\begin{aligned} \{F, G\}_\lambda &= \{F, G\}_2 - \lambda \{F, G\}_1 \\ &= \frac{1}{\epsilon} \text{Tr} \left( (LY)_+ LX - XL(YL)_+ + \frac{1}{n+1} X[L, g_Y] \right) - \lambda \frac{1}{\epsilon} \text{Tr} ([Y, X]L), \end{aligned} \quad (6.11)$$

where  $X = \frac{\delta F}{\delta L}$ ,  $Y = \frac{\delta G}{\delta L}$ , and the positive part of a pseudo-differential operator  $Z = \sum z_i D^i$  is defined by

$$Z_+ = \sum_{i \geq 0} z_i D^i.$$

The function  $g_Y$  is defined by

$$g_Y = D^{-1}(\text{res}[L, Y]).$$

Due to (6.10),  $g_Y$  is a differential polynomial of the coefficients of the operators  $L, Y$ .



In the computation of Poisson brackets of our type it suffices to deal with the linear functionals

$$\ell_X = \int \sum_{i=1}^n a_i(x) u_i(x) dx, \quad \ell_Y = \int \sum_{i=1}^n b_i(x) u_i(x) dx. \quad (6.12)$$

Then the operators  $X = \delta \ell_X / \delta L$ ,  $Y = \delta \ell_Y / \delta L$  read

$$X = \sum_{i=1}^n D^{-i} a_i(x), \quad Y = \sum_{i=1}^n D^{-i} b_i(x). \quad (6.13)$$

For a pseudo-differential operator  $Z = \sum_{i \leq m} z_i(x) D^i$ , define its symbol as

$$\hat{Z}(x, p) = \sum_{i \leq m} z_i(x) p^i.$$

The symbol of the composition of two pseudo-differential operators can be computed by the following well known formula<sup>4</sup>

$$\begin{aligned} \widehat{Z_1 Z_2}(x, p) &= \hat{Z}_1(x, p) \star \hat{Z}_2(x, p) := e^{\epsilon \frac{\partial^2}{\partial p \partial x'}} \hat{Z}_1(x, p) \hat{Z}_2(x', p')|_{x'=x, p'=p} \\ &= \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \partial_p^k \hat{Z}_1(x, p) \partial_x^k \hat{Z}_2(x, p). \end{aligned} \quad (6.14)$$

Taking the commutator in the leading term one obtains the Poisson bracket on the  $(x, p)$ -plane as follows

$$\begin{aligned} f(x, p) \star g(x, p) - g(x, p) \star f(x, p) &= \epsilon \{f, g\} + O(\epsilon^2), \\ \{f, g\} &:= \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial g}{\partial p} \frac{\partial f}{\partial x}. \end{aligned} \quad (6.15)$$

In the sequel we will often omit writing explicitly the  $x$ -dependence of the symbol.

The symbol of the positive part of a pseudo-differential operator can be computed by Cauchy integral formula

$$\widehat{Z_+}(p) = \left( \hat{Z}(p) \right)_+ = \oint \frac{dq}{2\pi i} \frac{\hat{Z}(q)}{q - p}. \quad (6.16)$$

---

<sup>4</sup>Warning: we use here the symbol  $\star$  that usually arises in the quantization of Poisson brackets. However our “star product” is different from the standard one.

where the integration is taken along the circle of radius  $|q| > |p|$ .

Let

$$\lambda(x, p) = p^{n+1} + u_n(x)p^{n-1} + \cdots + u_2(x)p + u_1(x) = \hat{L}$$

be the symbol of the Lax operator (6.8).

**Theorem 6.1** (i) *The dispersionless limit of the  $A_n$  Drinfeld - Sokolov bi-hamiltonian structure is given by the following formulae*

$$\begin{aligned} \{\lambda(x, p), \lambda(y, q)\}_1 &= \frac{\lambda'(p) - \lambda'(q)}{p - q} \delta'(x - y) \\ &+ \left[ \frac{\lambda_x(p) - \lambda_x(q)}{(p - q)^2} - \frac{\lambda'_x(q)}{p - q} \right] \delta(x - y), \end{aligned} \quad (6.17)$$

$$\begin{aligned} &\{\lambda(x, p), \lambda(y, q)\}_2 \\ &= \left( \frac{\lambda'(p)\lambda(q) - \lambda'(q)\lambda(p)}{p - q} + \frac{1}{n+1} \lambda'(p)\lambda'(q) \right) \delta'(x - y) \\ &+ \left[ \frac{\lambda_x(p)\lambda(q) - \lambda_x(q)\lambda(p)}{(p - q)^2} + \frac{\lambda_x(q)\lambda'(p) - \lambda'_x(q)\lambda(p)}{p - q} \right. \\ &\quad \left. + \frac{1}{n+1} \lambda'(p)\lambda'_x(q) \right] \delta(x - y). \end{aligned} \quad (6.18)$$

(ii) *The central invariants of the bihamiltonian structure are equal to*

$$c_1 = c_2 = \cdots = c_n = \frac{1}{24}.$$

Before proceeding to the proof let us explain the notations in the formulae (6.17) - (6.18). In the left hand sides we simply write the generating polynomials for the matrices  $\{u_i(x), u_j(y)\}_{1,2}$  of Poisson brackets, i.e.,

$$\{\lambda(x, p), \lambda(y, q)\}_{1,2} = \sum_{i,j=1}^n \{u_i(x), u_j(y)\}_{1,2} p^{i-1} q^{j-1}.$$

In the right hand sides we denote  $\lambda(p) \equiv \lambda(x, p)$ ,

$$\lambda'(p) = \frac{\partial}{\partial p} \lambda(x, p), \quad \lambda_x(p) = \partial_x \lambda(x, p).$$

Same for the terms depending on  $q$ , i.e.  $\lambda(q) \equiv \lambda(x, q)$ ,  $\lambda'(q) = \frac{\partial}{\partial q} \lambda(x, q)$  etc. Observe that the sign of the second metric (the coefficients of  $\delta'(x - y)$  of (6.18)) is opposite to the one given in Proposition 2.4.2 of [49].

*Proof* Let us introduce the symbols

$$f(p) = \sum_{i=1}^n \frac{a_i(x)}{p^i}, \quad g(p) = \sum_{i=1}^n \frac{b_i(x)}{p^i}. \quad (6.19)$$

They are related to the symbols of the operators (6.13) via

$$\hat{X}(p) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \partial_p^k \partial_x^k f(p), \quad \hat{Y}(p) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \partial_p^k \partial_x^k g(p). \quad (6.20)$$

We begin with the calculation of the leading term of the first Poisson bracket. Due to (6.15) one obtains

$$\{\ell_X, \ell_Y\}_1 = \int \text{res}(\{g(x, p), f(x, p)\} \lambda(x, p)) dx + O(\epsilon).$$

Here  $\text{res}$  of a symbol is just the coefficient of  $p^{-1}$ . Integrating by parts one rewrites

$$\int \text{res}(\{g, f\} \lambda) dx = \int \text{res}(f \{\lambda, g\}) dx.$$

As the series  $f$  contains only negative powers of  $p$ , one can replace the series  $\{\lambda, g\}$  by its positive part

$$\{\lambda, g\}_+ = \oint \frac{dq}{2\pi i} \frac{\lambda'(q)g_x(q) - \lambda_x(q)g'(q)}{q - p}.$$

Integrating by parts in  $q$  and inserting two zero terms

$$- \oint \frac{dq}{2\pi i} \frac{\lambda'(p)}{q - p} g_x(q) = 0, \quad \oint \frac{dq}{2\pi i} \frac{\lambda_x(p)}{(q - p)^2} g(q) = 0$$

one obtains the following expression for the leading term of the first Poisson bracket

$$\begin{aligned} \{\ell_X, \ell_Y\}_1 &= \int dx \oint \frac{dp}{2\pi i} \oint \frac{dq}{2\pi i} \left[ f(p) \frac{\lambda'(p) - \lambda'(q)}{p - q} g_x(q) \right. \\ &\quad \left. + \left( \frac{\lambda_x(p) - \lambda_x(q)}{(p - q)^2} - \frac{\lambda'_x(q)}{p - q} \right) f(p) g(q) \right] + O(\epsilon). \end{aligned}$$

This gives the formula (6.17). Note that the rational functions

$$\frac{\lambda'(p) - \lambda'(q)}{p - q}$$

and

$$\frac{\lambda_x(p) - \lambda_x(q)}{(p - q)^2} - \frac{\lambda'_x(q)}{p - q}$$

have no singularity on the diagonal, so the order of the loop integrals is inessential.

A similar computation proves also the formula (6.18).

Let us proceed to computing the higher order corrections. Note that what we want to compute is just four tensors  $P_a^{ij}(u), Q_a^{ij}(u)$  ( $a = 1, 2$ ) independent of the jet coordinates (see (2.12)). So through the computation we can omit all the derivatives of  $u_i$  w.r.t.  $x$ , i.e. we can treat  $u_i$  as constants. By using this assumption, one can obtain

$$g_Y = \oint \frac{dq}{2\pi i} \sum_{k=1}^{\infty} \frac{\epsilon^{k-1}}{k!} \partial_p^k \hat{L}(q) \partial_x^{k-1} \hat{Y}(q). \quad (6.21)$$

By substituting the formulae (6.14), (6.16), (6.20), (6.21) into the formula (6.11), we can obtain

$$\{\ell_X, \ell_Y\}_a = \int dx \oint \frac{dp}{2\pi i} \oint \frac{dq}{2\pi i} \sum_{k,i,s,j,t \geq 0} \partial_p^i \partial_x^s f(p) \tilde{A}_{a,k,i,s,j,t}(p, q, x) \epsilon^{k+s+t-1} \partial_q^j \partial_x^t g(q), \quad a = 1, 2.$$

After few integration by parts, the above equation reduces to the following one

$$\{\ell_X, \ell_Y\}_a = \int dx \oint \frac{dp}{2\pi i} \oint \frac{dq}{2\pi i} \sum_{k,s \geq 0} f(p) A_{a,k,s}(p, q, x) \epsilon^{k+s-1} \partial_x^s g(q). \quad (6.22)$$

We already know the coefficients

$$A_{1,0,1} = \frac{\lambda'(p) - \lambda'(q)}{p - q}$$

and

$$A_{2,0,1} = \frac{\lambda'(p)\lambda(q) - \lambda'(q)\lambda(p)}{p - q} + \frac{1}{n+1} \lambda'(p)\lambda'(q).$$

The subsequent coefficients  $A_{a,0,2}$ ,  $A_{a,0,3}$  ( $a = 1, 2$ ) read

$$\begin{aligned}
A_{1,0,2} &= \frac{\lambda'(q) - \lambda'(p)}{(q-p)^2} - \frac{\lambda''(q) + \lambda''(p)}{2(q-p)}, \\
A_{1,0,3} &= \frac{\lambda'(q) - \lambda'(p)}{(q-p)^3} - \frac{\lambda''(q) + \lambda''(p)}{2(q-p)^2} + \frac{\lambda'''(q) - \lambda'''(p)}{6(q-p)}, \\
A_{2,0,2} &= \frac{\lambda'(q)\lambda(p) - \lambda(q)\lambda'(p)}{(q-p)^2} - \frac{\lambda''(q)\lambda(p) - 2\lambda'(q)\lambda'(p) + \lambda(q)\lambda''(p)}{2(q-p)} \\
&\quad - \frac{\lambda''(q)\lambda'(p) - \lambda'(q)\lambda''(p)}{2(n+1)}, \\
A_{2,0,3} &= \frac{\lambda'(q)\lambda(p) - \lambda(q)\lambda'(p)}{(q-p)^3} - \frac{\lambda''(q)\lambda(p) - 2\lambda'(q)\lambda'(p) + \lambda(q)\lambda''(p)}{2(q-p)^2} \\
&\quad + \frac{\lambda'''(q)\lambda(p) - 3\lambda''(q)\lambda'(p) + 3\lambda'(q)\lambda''(p) - \lambda(q)\lambda'''(p)}{6(q-p)} \\
&\quad + \frac{2\lambda'''(q)\lambda'(p) - 3\lambda''(q)\lambda''(p) + 2\lambda'(q)\lambda'''(p)}{12(n+1)}. \tag{6.23}
\end{aligned}$$

Now we introduce two complex numbers  $P, Q$  such that  $|P| < |p|, |Q| < |q|$ , and define the functions  $f(p), g(p)$  as

$$f(p) = \frac{1}{p-P}\delta(x-y) = \sum_{i=1}^{\infty} \frac{P^{i-1}}{p^i}\delta(x-y), \quad g(p) = \frac{1}{q-Q}\delta(x-z).$$

Here, unlike the form given in (6.19), we allow the symbols  $f(p), g(p)$  to contain terms of the form  $\frac{1}{p^i}$  with  $i > n$ . However, it is easy to see that these additional terms do not affect the Poisson bracket (6.22).

It follows then that

$$\begin{aligned}
\ell_X &= \lambda(y, P) - P^{n+1} = u_n(y)P^{n-1} + \cdots + u_2(y)P + u_1(y), \\
\ell_Y &= \lambda(z, Q) - Q^{n+1} = u_n(z)Q^{n-1} + \cdots + u_2(z)Q + u_1(z),
\end{aligned}$$

and the formula (6.22) reads

$$\begin{aligned}
\{\lambda(y, P), \lambda(z, Q)\}_a &= \sum_{k,s \geq 0} \epsilon^{k+s-1} \delta^{(s)}(y-z) \left[ \oint \frac{dp}{2\pi i} \oint \frac{dq}{2\pi i} \frac{A_{a,k,s}(p, q, y)}{(p-P)(q-Q)} \right] \\
&= \sum_{k,s \geq 0} \epsilon^{k+s-1} A_{a,k,s}(P, Q, y) \delta^{(s)}(y-z). \tag{6.24}
\end{aligned}$$

Let  $r_1, \dots, r_n$  be the critical points of the polynomial  $\lambda(p)$ , i.e., the roots of  $\lambda'(r) = 0$ . Assuming them to be pairwise distinct, we have

$$A_{1,0,1}(r_i, r_j, x) = \delta_{ij} \lambda''(x, r_i), \quad A_{2,0,1}(r_i, r_j, x) = \delta_{ij} \lambda(x, r_i) \lambda''(x, r_i).$$

This shows that the critical *values*  $\lambda_i = \lambda(r_i)$  are the canonical coordinates of the bihamiltonian structure (6.24). Then the quantities in the formula (2.12) read

$$\begin{aligned} f^i &= \lambda''(r_i), \\ Q_1^{ii} &= \frac{1}{12} \lambda^{(4)}(r_i), \quad Q_2^{ii} = \frac{1}{12} \lambda(r_i) \lambda^{(4)}(r_i) + \frac{n}{n+1} \frac{\lambda''(r_i)^2}{4}, \\ P_1^{ki} &= \frac{\lambda''(r_k) + \lambda''(r_i)}{2(r_k - r_i)}, \quad P_2^{ki} = \frac{\lambda''(r_k) \lambda(r_i) + \lambda(r_k) \lambda''(r_i)}{2(r_k - r_i)}. \end{aligned}$$

Thus the central invariants read

$$\begin{aligned} c_i &= \frac{1}{3\lambda''(r_i)^2} \left( \frac{n}{n+1} \frac{\lambda''(r_i)^2}{4} + \sum_{k \neq i} \frac{(\lambda(r_k) - \lambda(r_i)) \lambda''(r_i)^2}{4\lambda''(r_k)(r_k - r_i)^2} \right) \\ &= \frac{1}{12} \left( \frac{n}{n+1} + \sum_{k \neq i} \frac{(\lambda(r_k) - \lambda(r_i))}{\lambda''(r_k)(r_k - r_i)^2} \right) = \frac{1}{12} \left( \frac{n}{n+1} + \frac{1-n}{2(n+1)} \right) \\ &= \frac{1}{24}. \end{aligned}$$

Here the third equality is obtained by applying the residue theorem to the meromorphic function

$$m(q) = \frac{\lambda(q) - \lambda(r_i)}{\lambda'(q)(q - r_i)^2}.$$

The Theorem is proved. □

## 7 The $B_n$ , $C_n$ and $D_n$ cases

The simple algebras of type  $B_n$ ,  $C_n$  and  $D_n$  can be realized as matrix Lie algebras  $o(2n+1)$ ,  $sp(2n)$  and  $o(2n)$ . The details of these realizations are omitted here, see Appendix 1 of [13]. Note that the Weyl generators  $X_i, Y_i, H_i$  we choose here correspond respectively to  $Y_i, X_i, -H_i$  of [13]. We begin with the following scalar differential operators satisfying certain sym-

metry/antisymmetry conditions:

$$B_n : \quad L = D^{2n+1} + \sum_{i=1}^n u_i(x) D^{2i-1} + \sum_{i=1}^n v_i(x) D^{2i-2}, \quad L + L^\dagger = 0 \quad (7.1)$$

$$C_n : \quad L = D^{2n} + \sum_{i=1}^n u_i(x) D^{2i-2} + \sum_{i=2}^n v_i(x) D^{2i-3}, \quad L = L^\dagger \quad (7.2)$$

$$D_n : \quad L = D^{2n-1} + \sum_{i=2}^n u_i(x) D^{2i-3} + \sum_{i=2}^n v_i(x) D^{2i-4} + \rho(x) D^{-1} \rho(x), \quad (7.3)$$

$$L + L^\dagger = 0.$$

Here  $L^\dagger$  is the adjoint operator (2.7), the coefficients  $v_i(x)$  are linear combinations of derivatives of  $u_i(x)$  uniquely determined by the symmetry/antisymmetry conditions. We assume  $u_1(x) = \rho^2(x)$  for the  $D_n$  case.

As for the  $A_n$  case, the above scalar (pseudo) differential operators can also be derived from the differential operator  $\mathcal{L}$  of the form (6.4). In the present cases, the matrices  $q$  are upper triangular ones belonging to  $\mathfrak{o}(2n+1)$ ,  $\mathfrak{sp}(2n)$  and  $\mathfrak{o}(2n)$  respectively. The matrices  $I$  are given respectively by

$$I = \sum_{i=1}^n (e_{i+1,i} + e_{2n+2-i,2n+1-i}), \quad I = \sum_{i=1}^{n-1} (e_{i+1,i} + e_{2n+1-i,2n-i}) + e_{n+1,n}$$

and

$$I = \sum_{i=1}^{n-1} (e_{i+1,i} + e_{2n+1-i,2n-i}) + \frac{1}{2}(e_{n+1,n-1} + e_{n+2,n}).$$

The scalar differential operators  $L$  are given by  $-\Delta(\mathcal{L})^\dagger$ , where the operator  $\Delta$  is defined as in (6.6).

The variational derivative of a functional of  $L$  w.r.t.  $L$  is now defined as

$$\frac{\delta F}{\delta L} = \frac{1}{2} \sum_{i=1}^n \left( D^{-2i+\nu} \frac{\delta F}{\delta u_i(x)} + \frac{\delta F}{\delta u_i(x)} D^{-2i+\nu} \right), \quad (7.4)$$

where  $\nu = 0, 1, 2$  for the  $B_n$ ,  $C_n$  and  $D_n$  cases respectively. This definition ensures the validity of (6.9).

In order to have a uniform expression of the Drinfeld - Sokolov second hamiltonian structures for the three types of simple Lie algebras, we fix in this section the invariant bilinear form on  $\mathfrak{g}$  by

$$\langle a, b \rangle_{\mathfrak{g}} = \text{tr}(ab). \quad (7.5)$$

Let us note that the normalized invariant bilinear form defined in (5.26) for the simple Lie algebras of type  $B_n, C_n, D_n$  have the expressions

$$\frac{1}{2} \operatorname{tr}(ab), \quad \operatorname{tr}(ab), \quad \frac{1}{2} \operatorname{tr}(ab) \quad (7.6)$$

respectively. With the above fixed invariant bilinear form, the second hamiltonian structures for the three types of simple Lie algebras have a uniform expression

$$\{F, G\}_2 = \frac{1}{\epsilon} \operatorname{Tr} [(LY)_+ LX - XL(YL)_+], \quad (7.7)$$

while the first ones are defined as the Lie derivatives of the second ones along the coordinate  $u_i$ , where  $i = 1$  for  $B_n, C_n$  and  $i = 2$  for  $D_n$ ,

$$\{F, G\}_2(u_i, \dots) - \lambda \{F, G\}_1(u_i, \dots) = \{F, G\}_2(u_i - \lambda, \dots).$$

Explicitly,

$$B_n : \quad \{F, G\}_1 = \frac{1}{\epsilon} \operatorname{Tr} L(YDX - XDY), \quad (7.8)$$

$$C_n : \quad \{F, G\}_1 = \frac{1}{\epsilon} \operatorname{Tr} L(YX - XY), \quad (7.9)$$

$$D_n : \quad \{F, G\}_1 = \frac{1}{\epsilon} \operatorname{Tr} L(X_+DY_+ - Y_+DX_+ + Y_-DX_- - X_-DY_-). \quad (7.10)$$

Let us now describe the main result of this section. Let

$$\lambda_B(p) = p^{2n+1} + \sum_{i=1}^n u_i(x) p^{2i-1} \quad (7.11)$$

$$\lambda_C(p) = p^{2n} + \sum_{i=1}^n u_i(x) p^{2i-2} \quad (7.12)$$

$$\lambda_D(p) = p^{2n-1} + \sum_{i=2}^n u_i(x) p^{2i-3} + \frac{u_1(x)}{p} \quad (7.13)$$

be the  $\epsilon = 0$  limits of the symbols of the Lax operators (7.1) - (7.3). Introduce

$$\begin{aligned} \Lambda_B(P) &= \Lambda_C(P) = P^n + u_n(x) P^{n-1} + \dots + u_1(x) \\ \Lambda_D(P) &= P^{n-1} + u_n(x) P^{n-1} + \dots + u_{n-1}(x) + \frac{u_1(x)}{P}. \end{aligned} \quad (7.14)$$

by the following substitution:

$$\begin{aligned} \lambda_B(p) &= p \Lambda_B(p^2) \\ \lambda_C(p) &= \Lambda_C(p^2) \\ \lambda_D(p) &= p \Lambda_D(p^2). \end{aligned} \quad (7.15)$$



**Theorem 7.1** (i) *The dispersionless limits of the Drinfeld - Sokolov bihamiltonian structures associated to the simple Lie algebras of type  $B_n$ ,  $C_n$ , and  $D_n$  have the following uniform expression*

$$\begin{aligned} \{\Lambda(x, P), \Lambda(y, Q)\}_1 &= 2 \frac{P\Lambda'(P) - Q\Lambda'(Q)}{P - Q} \delta'(x - y) \\ &+ \left[ \frac{P + Q}{(P - Q)^2} (\Lambda_x(P) - \Lambda_x(Q)) - 2 \frac{Q\Lambda'_x(Q)}{P - Q} \right] \delta(x - y), \end{aligned} \quad (7.16)$$

$$\begin{aligned} \{\Lambda(x, P), \Lambda(y, Q)\}_2 &= 2 \frac{P\Lambda'(P)\Lambda(Q) - Q\Lambda'(Q)\Lambda(P)}{P - Q} \delta'(x - y) \\ &+ \left[ \frac{P + Q}{(P - Q)^2} (\Lambda_x(P)\Lambda(Q) - \Lambda_x(Q)\Lambda(P)) \right. \\ &\quad \left. + 2 \frac{P\Lambda'(P)\Lambda_x(Q) - Q\Lambda'_x(Q)\Lambda(P)}{P - Q} \right] \delta(x - y), \end{aligned} \quad (7.17)$$

where  $\Lambda(x, P) = \Lambda_B, \Lambda_C$ , or  $\Lambda_D$  respectively.

(ii) *The central invariants of the Drinfeld - Sokolov bihamiltonian structures read*

$$B_n : \quad c_1 = \cdots = c_{n-1} = \frac{1}{12}, \quad c_n = \frac{1}{6} \quad (7.18)$$

$$C_n : \quad c_1 = \cdots = c_{n-1} = \frac{1}{12}, \quad c_n = \frac{1}{24} \quad (7.19)$$

$$D_n : \quad c_1 = c_2 = \cdots = c_n = \frac{1}{12}. \quad (7.20)$$

Note that the rescaling

$$\langle \cdot, \cdot \rangle_{\mathfrak{g}} \mapsto \kappa \langle \cdot, \cdot \rangle_{\mathfrak{g}}$$

of the invariant bilinear form on  $\mathfrak{g}$  yields the rescaling of the central invariants (2.12) of the related Drinfeld - Sokolov bihamiltonian structure

$$c_i \mapsto \kappa c_i, \quad i = 1, \dots, n.$$

So from the definition of the normalized bilinear form (5.26) and (6.2), (7.5), (7.6) and Theorem 6.1, Theorem 7.1 it follows the validity of the Theorem 5.5 for the cases  $A_n, B_n, C_n, D_n$ .

Before proceeding to the proof of the Theorem let us explain the rule of labeling of the central invariants for the  $B_n$  and  $C_n$  cases. The reader may remember that the labeling of the central invariants is in one-to-one

correspondence with labeling of the canonical coordinates. It will be shown below that the canonical coordinates for the bihamiltonian structure (7.16), (7.17) are defined as follows:

$$\begin{aligned}\lambda_i &= \Lambda(r_i^2), \quad i = 1, \dots, n \\ \frac{d}{dp} \Lambda(p^2)|_{p=r_i} &= 0.\end{aligned}\tag{7.21}$$

For  $B_n$  and  $C_n$  cases  $r_n = 0$  is always a critical point of  $\Lambda(p^2)$ . The associated critical value  $\lambda_n = \Lambda(0)$  “breaks the symmetry” between the canonical coordinates; the corresponding central invariant  $c_n$  differs from others.

*Proof* The derivation of the dispersionless Poisson structures (7.16), (7.17) follows the lines of the proof of Theorem 6.1. We will omit this part of the proof, and proceed directly to computation of the central invariants.

Some part of the computation can be done uniformly for all the three types of Lie algebras. To this end we introduce the symbol

$$\lambda(p) = p^{2n+1-\nu} + \sum_{i=1}^n u_i(x) p^{2i-1-\nu}\tag{7.22}$$

and also

$$f(p) = \sum_{i \geq 1} \frac{a_i(x)}{p^{2i-\nu}}, \quad g(p) = \sum_{i \geq 1} \frac{b_i(x)}{p^{2i-\nu}}\tag{7.23}$$

(we plan to still use the linear functionals (6.12)). Recall that  $\nu = 0, 1, 2$  for  $B_n, C_n$  and  $D_n$  respectively. The symbols of the pseudo-differential operators  $X$  and  $Y$  read

$$\hat{X}(p) = f(p) + \frac{1}{2} \sum_{k \geq 1} \frac{\epsilon^k}{k!} \partial_p^k \partial_x^k f(p), \quad \hat{Y}(p) = g(p) + \frac{1}{2} \sum_{k \geq 1} \frac{\epsilon^k}{k!} \partial_p^k \partial_x^k g(p).\tag{7.24}$$

We omit the derivatives of  $u_i$  w.r.t.  $x$  in  $\hat{L}(p)$  just like in the previous section.

By using the same method used in the proof of Theorem 6.1, we obtain the coefficients  $A_{2,0,1}$ ,  $A_{2,0,2}$  and  $A_{2,0,3}$  in the expansion (6.22)

$$\begin{aligned}A_{2,0,1} &= \frac{\lambda'(q)\lambda(p) - \lambda'(p)\lambda(q)}{q-p}, \quad A_{2,0,2} = 0, \\ A_{2,0,3} &= \frac{\lambda'(q)\lambda(p) - \lambda'(p)\lambda(q)}{2(q-p)^3} - \frac{\lambda''(q)\lambda(p) - 2\lambda'(q)\lambda'(p) + \lambda''(p)\lambda(q)}{4(q-p)^2} \\ &\quad + \frac{\lambda'(q)\lambda''(p) - \lambda'(p)\lambda''(q)}{4(q-p)} + \frac{\lambda'''(q)\lambda(p) - \lambda'''(p)\lambda(q)}{6(q-p)}.\end{aligned}\tag{7.25}$$

Now let  $P, Q$  be two complex numbers such that  $|P| < |p|^2$  and  $|Q| < |q|^2$ . Define the functions  $a_i(x), b_i(x)$  as in (7.23) from the following expansions

$$f(p) = \frac{p^\nu}{p^2 - P} \delta(x - y) = \sum_{k=1}^{\infty} \frac{P^{k-1}}{p^{2k-\nu}} \delta(x - y), \quad g(q) = \frac{q^\nu}{q^2 - Q} \delta(x - z).$$

Then  $\ell_X = \Lambda(y, P) - P^n$ ,  $\ell_Y = \Lambda(z, Q) - Q^n$ , where

$$\Lambda(y, P) = P^n + u_n(y)P^{n-1} + \cdots + u_1(y). \quad (7.26)$$

The second Poisson bracket between the linear functionals now read

$$\{\Lambda(y, P), \Lambda(z, Q)\}_2 = \sum_{k,s \geq 0} \epsilon^{k+s-1} \delta^{(s)}(y-z) \left[ \oint \frac{dp}{2\pi i} \oint \frac{dp}{2\pi i} \frac{(pq)^\nu A_{2,k,s}(p, q, y)}{(p^2 - P)(q^2 - Q)} \right].$$

Denote by  $R_{2,1}$  the coefficient of  $\epsilon^0 \delta'(y - z)$ . It is easy to obtain

$$R_{2,1} = 2 \frac{P \Lambda'(P) \Lambda(Q) - Q \Lambda'(Q) \Lambda(P)}{P - Q}. \quad (7.27)$$

Here  $\Lambda(P) = \Lambda(y, P)$ ,  $\Lambda(Q) = \Lambda(z, Q)$ , and the primes stand for differentiations w.r.t.  $P$  or  $Q$ . Then by definition one can obtain the coefficient of  $\epsilon^0 \delta'(y - z)$  in  $\{\Lambda(y, P), \Lambda(z, Q)\}_1$  denoted by  $R_{1,1}$

$$B_n, C_n : \quad R_{1,1} = 2 \frac{P \Lambda'(P) - Q \Lambda'(Q)}{P - Q}, \quad (7.28)$$

$$D_n : \quad R_{1,1} = 2 \frac{P Q (\Lambda'(P) - \Lambda'(Q)) + P \Lambda(Q) - Q \Lambda(P)}{P - Q}. \quad (7.29)$$

Denote the coefficients of  $\epsilon^2 \delta'''(y - z)$  in  $\{\Lambda(y, P), \Lambda(z, Q)\}_\alpha$  by  $R_{\alpha,3}$ . After a lengthy computation, we obtain

$$\begin{aligned} B_n : \quad R_{2,3} = & \frac{(P + Q)^2 (\Lambda'(P) \Lambda(Q) - \Lambda'(Q) \Lambda(P))}{(P - Q)^3} + 4 \frac{P^2 \Lambda'''(P) \Lambda(Q) - Q^2 \Lambda'''(Q) \Lambda(P)}{3(P - Q)} \\ & + 2 \frac{P Q (\Lambda'(P) \Lambda''(Q) - \Lambda'(Q) \Lambda''(P))}{P - Q} + 2 \frac{P \Lambda''(P) \Lambda(Q) - Q \Lambda''(Q) \Lambda(P)}{P - Q} \\ & + 3 \Lambda'(P) \Lambda'(Q) - 2 \frac{P Q (\Lambda''(P) \Lambda(Q) - 2 \Lambda'(P) \Lambda'(Q) + \Lambda(P) \Lambda''(Q))}{(P - Q)^2}. \end{aligned} \quad (7.30)$$

$$\begin{aligned} R_{1,3} = & \frac{(P + Q)^2 (\Lambda'(P) - \Lambda'(Q))}{(P - Q)^3} + 4 \frac{P^2 \Lambda'''(P) - Q^2 \Lambda'''(Q)}{3(P - Q)} \\ & + 2 \frac{P \Lambda''(P) - Q \Lambda''(Q)}{P - Q} - 2 \frac{P Q (\Lambda''(P) + \Lambda''(Q))}{(P - Q)^2}. \end{aligned} \quad (7.31)$$

$$\begin{aligned}
C_n: \quad R_{2,3} = & \frac{(P^2 + 6PQ + Q^2)(\Lambda'(P)\Lambda(Q) - \Lambda'(Q)\Lambda(P))}{2(P-Q)^3} \\
& + 4\frac{P^2\Lambda'''(P)\Lambda(Q) - Q^2\Lambda'''(Q)\Lambda(P)}{3(P-Q)} + 2\frac{PQ(\Lambda'(P)\Lambda''(Q) - \Lambda'(Q)\Lambda''(P))}{P-Q} \\
& + \frac{P\Lambda''(P)\Lambda(Q) - Q\Lambda''(Q)\Lambda(P)}{P-Q} + \Lambda'(P)\Lambda'(Q) \\
& - 2\frac{PQ(\Lambda''(P)\Lambda(Q) - 2\Lambda'(P)\Lambda'(Q) + \Lambda(P)\Lambda''(Q))}{(P-Q)^2}. \tag{7.32}
\end{aligned}$$

$$\begin{aligned}
R_{1,3} = & \frac{(P^2 + 6PQ + Q^2)(\Lambda'(P) - \Lambda'(Q))}{2(P-Q)^3} + 4\frac{P^2\Lambda'''(P) - Q^2\Lambda'''(Q)}{3(P-Q)} \\
& + \frac{P\Lambda''(P) - Q\Lambda''(Q)}{P-Q} - 2\frac{PQ(\Lambda''(P) + \Lambda''(Q))}{(P-Q)^2}. \tag{7.33}
\end{aligned}$$

$$\begin{aligned}
D_n: \quad R_{2,3} = & 4\frac{PQ(\Lambda'(P)\Lambda(Q) - \Lambda'(Q)\Lambda(P))}{(P-Q)^3} + \\
& + 4\frac{P^2\Lambda'''(P)\Lambda(Q) - Q^2\Lambda'''(Q)\Lambda(P)}{3(P-Q)} + 2\frac{PQ(\Lambda'(P)\Lambda''(Q) - \Lambda'(Q)\Lambda''(P))}{P-Q} \\
& - \Lambda'(P)\Lambda'(Q) - 2\frac{PQ(\Lambda''(P)\Lambda(Q) - 2\Lambda'(P)\Lambda'(Q) + \Lambda(P)\Lambda''(Q))}{(P-Q)^2} \\
& + \frac{P^2\Lambda'(P)\Lambda(Q) - Q^2\Lambda'(Q)\Lambda(P)}{PQ(P-Q)} - \Lambda(0)\frac{P\Lambda'(P) + Q\Lambda'(Q)}{PQ}. \tag{7.34}
\end{aligned}$$

$$\begin{aligned}
R_{1,3} = & \frac{4PQ(P\Lambda'''(P) - Q\Lambda'''(Q))}{3(P-Q)} - 2\frac{PQ(P\Lambda''(P) + Q\Lambda''(Q))}{(P-Q)^2} \\
& + \frac{4PQ(P^2\tilde{\Lambda}'(P) - Q^2\tilde{\Lambda}'(Q))}{(P-Q)^3} + \frac{PQ(\tilde{\Lambda}'(P) - \tilde{\Lambda}'(Q))}{P-Q} - \frac{\Lambda(0)(P+Q)}{PQ}, \tag{7.35}
\end{aligned}$$

where

$$\tilde{\Lambda}(P) = \Lambda(P)/P. \tag{7.36}$$

Now we begin to compute the central invariants for the  $B_n, C_n$  cases. The formulae (7.27) (7.28) show that in these two cases we have the same dispersionless limit, so the corresponding Drinfeld - Sokolov bihamiltonian structures have the same canonical coordinates. Let  $r_1, \dots, r_n$  be defined as in (7.21). Then we have  $\lambda_n = u_1$  and  $\lambda_1, \dots, \lambda_{n-1}$  are the critical values of  $\Lambda(P)$ . From the formulae (7.27) and (7.28), one can see that  $\lambda_1, \dots, \lambda_n$  can

serve as the canonical coordinates of the Drinfeld - Sokolov bihamiltonian structures of  $B_n$  and  $C_n$  type. Following the notations in (2.12), we have

$$B_n : \quad f^i = 2 r_i \Lambda''(r_i), \quad f^n = 2 \Lambda'(0);$$

$$\begin{aligned} Q_1^{ii} &= 3 \Lambda''(r_i) + \frac{14}{3} r_i \Lambda'''(r_i) + r_i^2 \Lambda''''(r_i), \quad Q_1^{nn} = 3 \Lambda''(0). \\ Q_2^{ii} &= r_i^2 \Lambda''(r_i)^2 + \Lambda(r_i) Q_1^{ii}, \quad Q_2^{nn} = 2 \Lambda'(0)^2 + 3 \Lambda(0) \Lambda''(0); \\ c_i &= \frac{Q_2^{ii} - \Lambda(r_i) Q_1^{ii}}{3(f^i)^2} = \frac{1}{12}, \quad c_n = \frac{Q_2^{nn} - \Lambda(0) Q_1^{nn}}{3(f^1)^2} = \frac{1}{6}. \end{aligned}$$

$$C_n : \quad f^i = 2 r_i \Lambda''(r_i), \quad f^n = 2 \Lambda'(0)$$

$$\begin{aligned} Q_1^{ii} &= 3 \Lambda''(r_i) + \frac{11}{3} r_i \Lambda'''(r_i) + r_i^2 \Lambda''''(r_i), \quad Q_1^{nn} = \frac{3}{2} \Lambda''(0); \\ Q_2^{ii} &= r_i^2 \Lambda''(r_i)^2 + \Lambda(r_i) Q_1^{ii}, \quad Q_2^{nn} = \frac{1}{2} \Lambda'(0)^2 + \frac{3}{2} \Lambda(0) \Lambda''(0); \\ c_i &= \frac{Q_2^{ii} - \Lambda(r_i) Q_1^{ii}}{3(f^i)^2} = \frac{1}{12}, \quad c_n = \frac{Q_2^{nn} - \Lambda(0) Q_1^{nn}}{3(f^1)^2} = \frac{1}{24}. \end{aligned}$$

Here  $i = 1, \dots, n$ .

To compute the central invariants for the  $D_n$  case, we first rewrite the two Poisson brackets in terms of the symbol  $\tilde{\Lambda}$  defined in (7.36). Let  $\tilde{R}_{\alpha,k}$  be obtained from  $R_{\alpha,k}$  of (7.30), (7.31) with  $\Lambda$  replaced by  $\tilde{\Lambda}$ . Denote by  $S_{\alpha,k}$  the coefficients of  $\epsilon^{k-1} \delta^{(k)}(y-z)$  in  $\{\tilde{\Lambda}(P, y), \tilde{\Lambda}(Q, z)\}_\alpha$ . Then we have  $S_{\alpha,1} = \tilde{R}_{\alpha,1}$ , and

$$S_{2,3} = \tilde{R}_{2,3} - \Lambda(0) \frac{P \Lambda'(P) + Q \Lambda'(Q)}{P^2 Q^2}, \quad S_{1,3} = \tilde{R}_{2,3} - \frac{\Lambda(0)(P+Q)}{P^2 Q^2}.$$

Let  $\lambda_1, \dots, \lambda_n$  be defined by (7.21), they are the critical values of the rational function  $\tilde{\Lambda}(P)$  and can serve as the canonical coordinates of the Drinfeld - Sokolov bihamiltonian structure in the  $D_n$  case. So we have

$$\begin{aligned} D_n : \quad f^i &= 2 r_i \tilde{\Lambda}''(r_i), \quad Q_1^{ii} = 3 \tilde{\Lambda}''(r_i) + \frac{14}{3} r_i \tilde{\Lambda}'''(r_i) + r_i^2 \tilde{\Lambda}''''(r_i) - \frac{2 \Lambda(0)}{r_i^3}, \\ Q_2^{ii} &= \tilde{\Lambda}(r_i) Q_1^{ii} + r_i^2 \tilde{\Lambda}''(r_i)^2, \quad c_i = \frac{Q_2^{ii} - \tilde{\Lambda}(r_i) Q_1^{ii}}{3(f^i)^2} = \frac{1}{12}. \end{aligned}$$

The Theorem is proved.  $\square$

## 8 The exceptional root systems

The hierarchies associated with the exceptional root systems have been systematically treated by V.Kac and M.Wakimoto in [34]. They did not consider however the bihamiltonian structure of the exceptional hierarchies. In our computations we will use the approach of [2] based on the Dirac reduction procedure [12]. Let us consider the Poisson bracket  $\pi_{\mathfrak{g}}(I)$  on  $\mathfrak{g}^*$  evaluated at the point  $I$  as a skew symmetric bilinear form on

$$\mathfrak{g} \simeq T_I^* \mathfrak{g}^*$$

(cf. (3.9)). The stabilizer  $\text{Ker ad}_I$  of  $I$  coincides with the kernel of this bilinear form. The quotient

$$\mathfrak{g}/\text{Ker ad}_I$$

acquires a symplectic structure induced by  $\pi_{\mathfrak{g}}(I)$ . The projection

$$\mathfrak{n} \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\text{Ker ad}_I$$

realizes the nilpotent subalgebra  $\mathfrak{n}$  as a Lagrangian subspace in the quotient. Let

$$\mathfrak{n}_{\text{dual}} \subset \mathfrak{h} \oplus \mathfrak{n}^- \tag{8.1}$$

be a pullback of a complementary Lagrangian subspace of the image of  $\mathfrak{n}$  such that

$$\mathfrak{g} = \text{Ker ad}_I \oplus \mathfrak{n} \oplus \mathfrak{n}_{\text{dual}}. \tag{8.2}$$

A choice of  $\mathfrak{n}_{\text{dual}}$  specifies the transversal subspace  $V \subset \mathfrak{b}$  of (4.19) by the equation

$$\langle b, q^{\text{can}} \rangle_{\mathfrak{g}} = 0 \quad \forall b \in \mathfrak{n}_{\text{dual}}, \quad q^{\text{can}} \in V. \tag{8.3}$$

One can unify constraints (4.18) and (8.3) by considering a system of equations for  $q \in \mathfrak{g}$ :

$$\begin{aligned} \langle a, q \rangle_{\mathfrak{g}} &= \langle a, I \rangle_{\mathfrak{g}} \quad \forall a \in \mathfrak{n} \\ \langle b, q \rangle_{\mathfrak{g}} &= 0 \quad \forall b \in \mathfrak{n}_{\text{dual}}. \end{aligned} \tag{8.4}$$

The solution

$$q = I + q^{\text{can}}$$

determines the transversal slice  $V$ . The reduced Poisson bracket on  $q^{\text{can}}$ -valued loops can be obtained as follows. Let us choose a basis

$$f_1, \dots, f_{2m} \in \mathfrak{n} \oplus \mathfrak{n}_{\text{dual}}, \quad 2m = 2 \dim \mathfrak{n} = \dim \mathfrak{g} - n.$$

Introduce two  $2m \times 2m$  matrices

$$\begin{aligned} P &= (P_{ab}), \quad P_{ab} = -\langle I + q^{\text{can}}, [f_a, f_b] \rangle_{\mathfrak{g}}, \\ Q &= (Q_{ab}), \quad Q_{ab} = \langle f_a, f_b \rangle_{\mathfrak{g}}. \end{aligned} \tag{8.5}$$

By construction of  $\mathfrak{n}_{\text{dual}}$  the matrix

$$P|_{q^{\text{can}}=0} = \pi_{\mathfrak{g}}(I)|_{\mathfrak{n} \oplus \mathfrak{n}_{\text{dual}}}$$

does not degenerate. Consider matrix differential operator

$$M := P + Q \epsilon \partial_x \tag{8.6}$$

with coefficients depending on  $q^{\text{can}}$  (via  $P$ ). Note that the matrix of pairwise Poisson brackets of the constraints (8.4) is equal to

$$\{\langle f_a, q(x) \rangle_{\mathfrak{g}}, \langle f_b, q(y) \rangle_{\mathfrak{g}}\} = -\frac{1}{\epsilon} M_{ab} \delta(x - y).$$

The following statement was proved in [25].

**Lemma 8.1** *The inverse  $M^{-1}$  to (8.6) is a matrix valued differential operator of finite order with coefficients depending polynomially on  $q^{\text{can}}, q_x^{\text{can}}, \dots$*

Let

$$\gamma^1, \dots, \gamma^n \in \text{Ker ad } I \tag{8.7}$$

be a basis in the centralizer of  $I$ . Recall [35] that this centralizer is a commutative subalgebra in  $\mathfrak{n}^-$  having generators only in the degrees  $-m_1, \dots, -m_n$ ; the number of generators in the degree  $-m_k$  is equal to the multiplicity of the exponent  $m_k$ . The linear functions of  $q^{\text{can}} \in V$  given by

$$u^i = \langle \gamma^i, q^{\text{can}} \rangle_{\mathfrak{g}}, \quad i = 1, \dots, n \tag{8.8}$$

define a system of coordinates on  $V$ . Denote  $\gamma_1, \dots, \gamma_n$  the dual basis in  $V$ ,

$$\langle \gamma^i, \gamma_j \rangle_{\mathfrak{g}} = \delta_j^i, \quad \langle f_a, \gamma_i \rangle_{\mathfrak{g}} = 0, \quad i, j = 1, \dots, n, \quad a = 1, \dots, 2m \tag{8.9}$$

so

$$q^{\text{can}} = \sum_{i=1}^n u^i \gamma_i. \tag{8.10}$$

Introduce the  $n \times 2m$  matrix differential operator

$$N = (N_a^i) = (R_a^i + S_a^i \epsilon \partial_x), \quad N_a^i = \epsilon \langle \gamma^i, f_a \rangle_{\mathfrak{g}} \partial_x - \langle q^{\text{can}}, [\gamma^i, f_a] \rangle_{\mathfrak{g}}. \quad (8.11)$$

Denote  $N^\dagger$  the matrix of (formally) adjoint differential operators,

$$(N^\dagger)_i^a = N_a^{i\dagger}, \quad i = 1, \dots, n, \quad a = 1, \dots, 2m. \quad (8.12)$$

Then the matrix of the second reduced Poisson bracket is given by the formula

$$\{u^i(x), u^j(y)\}_2^{\text{red}} = -\frac{1}{\epsilon} (N M^{-1} N^\dagger)^{ij} \delta(x - y). \quad (8.13)$$

The first reduced bracket is given by a similar formula

$$\begin{aligned} & \{u^i(x), u^j(y)\}_1^{\text{red}} \\ &= \frac{1}{\epsilon} \left( N M^{-1} \tilde{M} M^{-1} N^\dagger + \tilde{N} M^{-1} N^\dagger + N M^{-1} \tilde{N}^\dagger \right)^{ij} \delta(x - y) \end{aligned} \quad (8.14)$$

where the  $n \times 2m$  and  $2m \times 2m$  matrices  $\tilde{N}_a^i$  and  $\tilde{M}_{ab}$  respectively are defined as follows:

$$\tilde{N}_a^i = \langle \alpha, [\gamma^i, f_a] \rangle_{\mathfrak{g}}, \quad \tilde{M}_{ab} = \langle \alpha, [f_a, f_b] \rangle_{\mathfrak{g}}, \quad (8.15)$$

where  $\alpha \in \mathfrak{n}$  is the generator of the center of  $\mathfrak{n}$  chosen above (see (4.22)). We will see below that the terms of order  $\epsilon^{-1}$  disappear from (8.13), (8.14).

Let us now explain how we compute the Frobenius structure and the central invariants using the formula (8.13). For the second metric  $g_2^{ij}$  one obtains

$$(g_2^{ij}(q^{\text{can}})) = R P^{-1} Q P^{-1} R^T - S P^{-1} R^T + R P^{-1} S^T \quad (8.16)$$

where  $R^T, S^T$  denotes their transposed matrices. The matrices  $(A_{1,0;2}^{ij})$  and  $(A_{2,0;2}^{ij})$  have the following form:

$$\begin{aligned} (A_{1,0;2}^{ij}(q^{\text{can}})) &= -R P^{-1} Q P^{-1} Q P^{-1} R^T - R P^{-1} Q P^{-1} S^T \\ &+ S P^{-1} Q P^{-1} R^T + S P^{-1} S^T, \end{aligned} \quad (8.17)$$

$$\begin{aligned} (A_{2,0;2}^{ij}(q^{\text{can}})) &= R P^{-1} Q P^{-1} Q P^{-1} Q P^{-1} R^T - R P^{-1} Q P^{-1} Q P^{-1} S^T \\ &+ S P^{-1} Q P^{-1} Q P^{-1} R^T + S P^{-1} Q P^{-1} S^T, \end{aligned} \quad (8.18)$$

where the matrices  $R = (R_a^i)$ ,  $S = (S_a^i)$  are defined in (8.11). Doing the shift

$$q^{\text{can}} \mapsto q^{\text{can}} + \lambda \alpha, \quad \alpha \in \{\text{the center of } \mathfrak{n}\} \quad (8.19)$$



one obtains in (8.16) - (8.18) linear functions in  $\lambda$ . The coefficients of  $\lambda$  of these functions give the matrices  $g_1^{ij}$ ,  $A_{1,0;1}^{ij}(q^{\text{can}})$  and  $A_{2,0;1}^{ij}(q^{\text{can}})$  respectively.

The dual bases  $\gamma^i \in \text{Ker ad } I$  and  $\gamma_i \in V$  can be chosen as follows. According to [35] the triple

$$I_- := I, \quad \rho = \sum_{i=1}^n \omega_i, \quad I_+ = \sum_{i=1}^n a_i X_i \quad (8.20)$$

defines an embedding of the  $sl_2$  Lie algebra into  $\mathfrak{g}$ ,

$$[I_+, I_-] = 2\rho, \quad [\rho, I_{\pm}] = \pm I_{\pm}. \quad (8.21)$$

Here  $\omega_1, \dots, \omega_n \in \mathfrak{h}$  are the fundamental weights, i.e. the basis dual to the basis of simple roots, and the integer coefficients  $a_1, \dots, a_n$  are defined from the decomposition

$$2\rho = \sum_{i=1}^n a_i H_i. \quad (8.22)$$

We put

$$V := \text{Ker ad } I_+. \quad (8.23)$$

We choose

$$\gamma_i \in \text{Ker ad } I_+ \cap \mathfrak{g}^{m_i}, \quad i = 1, \dots, n \quad (8.24)$$

$$\gamma^i \in \text{Ker ad } I_- \cap \mathfrak{g}^{-m_i}, \quad i = 1, \dots, n$$

For all exceptional Lie algebras the vectors  $\gamma_i$  and  $\gamma^i$  are determined uniquely up to normalization. We can normalize them in such a way that

$$\langle \gamma^i, \gamma_j \rangle_{\mathfrak{g}} = \delta_j^i.$$

Denote

$$p(z; q^{\text{can}}) = \det [g_2^{ij}(q^{\text{can}}) - z g_1^{ij}(q^{\text{can}})] \quad (8.25)$$

the characteristic polynomial of the pair of quadratic forms  $g_2^{ij}$ ,  $g_1^{ij}$ . The roots  $z^1(q^{\text{can}}), \dots, z^n(q^{\text{can}})$  will be used as the canonical coordinates of the pair of metrics: in these coordinates both metrics become diagonal:

$$\begin{aligned} \sum_{k,l=1}^n \left( \frac{\partial p(z; q^{\text{can}})}{\partial u^k} \right)_{z=z^i} \left( \frac{\partial p(z; q^{\text{can}})}{\partial u^l} \right)_{z=z^j} g_1^{kl}(q^{\text{can}}) &= 0, \quad i \neq j, \\ \sum_{k,l=1}^n \left( \frac{\partial p(z; q^{\text{can}})}{\partial u^k} \right)_{z=z^i} \left( \frac{\partial p(z; q^{\text{can}})}{\partial u^l} \right)_{z=z^j} g_2^{kl}(q^{\text{can}}) &= 0, \quad i \neq j. \end{aligned} \quad (8.26)$$

Here we used the implicit function theorem formula

$$\frac{\partial z^i(q^{\text{can}})}{\partial u^k} = - \left( \frac{1}{p'(z; q^{\text{can}})} \frac{\partial p(z; q^{\text{can}})}{\partial u^k} \right)_{z=z^i}, \quad p'(z; q^{\text{can}}) = \frac{\partial p(z; q^{\text{can}})}{\partial z}.$$

For the central invariants one obtains the following expressions:

$$c_i = \frac{1}{3} [p'(z^i; q^{\text{can}})]^2 \quad (8.27)$$

$$\times \frac{\sum_{k,l=1}^n \left( \frac{\partial p(z; q^{\text{can}})}{\partial u^k} \right) \left( \frac{\partial p(z; q^{\text{can}})}{\partial u^l} \right) (A_{2,0;2}^{kl}(q^{\text{can}}) - z A_{2,0;1}^{kl}(q^{\text{can}}))}{\left[ \sum_{k,l=1}^n \left( \frac{\partial p(z; q^{\text{can}})}{\partial u^k} \right) \left( \frac{\partial p(z; q^{\text{can}})}{\partial u^l} \right) g_1^{kl}(q^{\text{can}}) \right]^2} \Bigg|_{z=z^i}.$$

We will now consider the  $G_2$ ,  $F_4$  and  $E_6 - E_8$  cases. The computer supported calculations for these Lie algebras of types  $F$  and  $E$  use an explicit realization of the Chevalley basis [6]. The Chevalley bases we describe in this paper are generated by the computer algebra system GAP [29]. For convenience of the reader we give an explicit matrix realization of the bases choosing a faithful representation of the Lie algebra of the minimal dimension. The source program is as follows:

PARAMETERS:

X: The type of the Lie algebras  
r: The rank of the Lie algebras  
w: The fundamental weight with the minimum dimension  
f: The name of the file that stores the result

PROGRAM:

```
L:=SimpleLieAlgebra("X",r,Rationals);;
V:=HighestWeightModule(L,w);;
ll:=Basis(L);;
vv:=Basis(V);;
mm:=List(ll, x->MatrixOfAction(vv,x));;
PrintTo("f",mm);;
```

The parameter  $w$  takes the following values:

$$\begin{aligned} F_4 : & [1, 0, 0, 0] \\ E_6 : & [0, 0, 0, 0, 0, 1] \\ E_7 : & [0, 0, 0, 0, 0, 0, 1] \\ E_8 : & [0, 0, 0, 0, 0, 0, 0, 1]. \end{aligned}$$

For the Lie algebras of type E, the results of the above program are exactly what we need, while for that of type F, we have to modify the signs of some base elements in order to satisfy our normalizing rules.

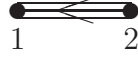
**Lemma 8.2** *For the exceptional Lie simple Lie algebras of type  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$ ,  $E_8$ , the central invariants of the corresponding Drinfeld - Sokolov bihamiltonian structures coincide with the values listed in the table that is given at the end of Section 5.*

*Proof* The lemma can be proved by a straightforward computation by using the formula (2.12) for the Lie algebras of  $G_2$  and  $F_4$  types. For the E type case we can use the formula (8.27) to compute the central invariants, however the computations become very involved; so we use a different method based on a comparison of the Drinfeld - Sokolov bihamiltonian structure with the one obtained in [18] (see below).

Since the central invariants do not depend on the choice of  $\alpha$  in (8.19), in what follows we will fix  $\alpha = \gamma_n$ .

We first illustrate the procedure by considering the  $G_2$  case<sup>5</sup> in detail.

Let  $X_i, H_i, Y_i$  ( $i = 1, 2$ ) be a set of Weyl generators of the simple Lie algebra  $\mathfrak{g}$  of  $G_2$  type, whose Dynkin diagram is labelled as follow



We define a Chevalley basis of  $\mathfrak{g}$

$$\begin{aligned} X_3 &= -[X_1, X_2], \quad Y_3 = [Y_1, Y_2], \\ X_4 &= -[X_1, X_3]/2, \quad Y_4 = [Y_1, Y_3]/2, \\ X_5 &= -[X_1, X_4]/3, \quad Y_5 = [Y_1, Y_4]/3, \\ X_6 &= -[X_2, X_5], \quad Y_6 = [Y_2, Y_5]. \end{aligned}$$

The normalized invariant bilinear form is given by

$$\begin{aligned} \langle X_1, Y_1 \rangle_{\mathfrak{g}} &= \langle X_3, Y_3 \rangle_{\mathfrak{g}} = \langle X_4, Y_4 \rangle_{\mathfrak{g}} = 3, \\ \langle X_2, Y_2 \rangle_{\mathfrak{g}} &= \langle X_5, Y_5 \rangle_{\mathfrak{g}} = \langle X_6, Y_6 \rangle_{\mathfrak{g}} = 1, \\ \langle H_1, H_1 \rangle_{\mathfrak{g}} &= 6, \quad \langle H_1, H_2 \rangle_{\mathfrak{g}} = -3, \quad \langle H_2, H_2 \rangle_{\mathfrak{g}} = 2. \end{aligned} \tag{8.28}$$

The elements  $\rho, I_+$  read

$$\rho = 3H_1 + 5H_2, \quad I_+ = 6X_1 + 10X_2.$$

---

<sup>5</sup>Explicit formulae for the  $G_2$  bihamiltonian structure were obtained in the original paper [13]. In [25] they have been rederived using the Dirac reduction procedures.

We choose a basis of  $\text{Ker ad } I_+$

$$\gamma_1 = \frac{3}{5}X_1 + X_2, \quad \gamma_2 = X_6.$$

Then we can obtain the result of the Dirac reduction:

$$\begin{aligned} (g_2^{ij}) &= \begin{pmatrix} -\frac{5u_1}{7} & -\frac{15u_2}{7} \\ -\frac{15u_2}{7} & -\frac{768u_1^5}{875} - \frac{1144}{525}u_2u_1^2 \end{pmatrix}, \quad (g_1^{ij}) = \begin{pmatrix} 0 & -\frac{15}{7} \\ -\frac{15}{7} & -\frac{1144u_1^2}{525} \end{pmatrix}, \\ (A_{2,0,2}^{ij}) &= \begin{pmatrix} \frac{25}{14} & 0 \\ 0 & \frac{42152u_1^4}{13125} + \frac{62u_2u_1}{21} \end{pmatrix}, \quad (A_{2,0,1}^{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{62u_1}{21} \end{pmatrix}, \end{aligned}$$

and  $A_{1,0,2}^{ij} = A_{1,0,1}^{ij} = 0$ .

If we introduce the flat coordinates

$$t_1 = u_2 - \frac{572u_1^3}{3375}, \quad t_2 = -\frac{7u_1}{15},$$

the above metrics are just the flat pencil defined by the following Frobenius manifold

$$F = \frac{1}{2}t_1^2t_2 + \frac{24}{35}t_2^7, \quad E = t_1\frac{\partial}{\partial t_1} + \frac{1}{3}t_2\frac{\partial}{\partial t_2}.$$

In the flat coordinates, we have

$$\begin{aligned} (g_2^{ij}) &= \begin{pmatrix} 48t_2^5 & t_1 \\ t_1 & \frac{t_2}{3} \end{pmatrix}, \quad (g_1^{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ (A_{2,0,2}^{ij}) &= \begin{pmatrix} -\frac{310t_2}{49} & 0 \\ 0 & 0 \end{pmatrix}, \quad (A_{2,0,1}^{ij}) = \begin{pmatrix} 88t_2^4 - \frac{310t_1t_2}{49} & \frac{286t_2^2}{147} \\ \frac{286t_2^2}{147} & \frac{7}{18} \end{pmatrix} \end{aligned}$$

The canonical coordinates are

$$\lambda_1 = t_1 + 4t_2^3, \quad \lambda_2 = t_1 - 4t_2^3,$$

from which we can compute the quantities appeared in the formula (2.12)

$$\begin{aligned} f^1 &= 24t_2^2, \quad f^2 = -24t_2^2, \\ Q_2^{11} &= \frac{9344t_2^4}{49} - \frac{310t_1t_2}{49}, \quad Q_2^{22} = \frac{4768t_2^4}{49} - \frac{310t_1t_2}{49}, \\ Q_1^{11} &= -\frac{310t_2}{49}, \quad Q_1^{22} = -\frac{310t_2}{49}. \end{aligned}$$

So the central invariants are given by

$$c_1 = \frac{Q_2^{11} - \lambda_1 Q_1^{11}}{3(f^1)^2} = \frac{1}{8}, \quad c_2 = \frac{Q_2^{22} - \lambda_2 Q_1^{22}}{3(f^2)^2} = \frac{1}{24}.$$

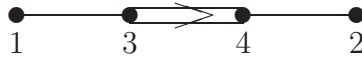
### The $F_4$ case

The root system of type  $F_4$  contains 24 positive roots, it's not convenient to define the Chevalley basis explicitly, so we use an alternative way below to describe this basis.

The simple Lie algebra of type  $F_4$  has a 26-dimensional matrix realization, whose Weyl generators are

$$\begin{aligned}
X_1 &= e_{1,2} + e_{6,8} + e_{7,10} + e_{9,12} + 2e_{11,13} + e_{11,14} \\
&\quad + e_{13,16} + e_{15,18} + e_{17,20} + e_{19,21} + e_{25,26}, \\
X_2 &= e_{4,5} + e_{6,7} + e_{8,10} - e_{17,19} - e_{20,21} - e_{22,23}, \\
X_3 &= e_{2,3} - e_{4,6} - e_{5,7} - e_{9,11} - e_{12,13} - 2e_{12,14} \\
&\quad - e_{14,15} - e_{16,18} + e_{20,22} + e_{21,23} - e_{24,25}, \\
X_4 &= e_{3,4} - e_{7,9} - e_{10,12} + e_{15,17} + e_{18,20} + e_{23,24}, \\
\\
Y_1 &= e_{2,1} + e_{8,6} + e_{10,7} + e_{12,9} + e_{13,11} + 2e_{16,13} \\
&\quad + e_{16,14} + e_{18,15} + e_{20,17} + e_{21,19} + e_{26,25}, \\
Y_2 &= e_{5,4} + e_{7,6} + e_{10,8} - e_{19,17} - e_{21,20} - e_{23,22}, \\
Y_3 &= e_{3,2} - e_{6,4} - e_{7,5} - e_{11,9} - e_{14,12} - e_{15,13} \\
&\quad - 2e_{15,14} - e_{18,16} + e_{22,20} + e_{23,21} - e_{25,24}, \\
Y_4 &= e_{4,3} - e_{9,7} - e_{12,10} + e_{17,15} + e_{20,18} + e_{24,23}.
\end{aligned}$$

These generators correspond the following Dynkin diagram



The normalized Killing form can be computed by the following formula

$$\langle A, B \rangle_{\mathfrak{g}} = \frac{1}{6} \text{tr}(A B).$$

Let  $\alpha_i$  be the simple root corresponding to  $X_i$ ,  $i = 1, \dots, 4$ . For any positive root  $\beta \in \Phi^+$  of the form

$$\beta = \sum_{i=1}^4 n_i \alpha_i, \text{ where } n_i \geq 0, \ i = 1, \dots, 4,$$

we define  $X_\beta = X_{n_1, \dots, n_4}$  (resp.  $Y_\beta = Y_{n_1, \dots, n_4}$ ) to be the matrix in the root space  $\mathfrak{g}_\beta$  (resp.  $\mathfrak{g}_{-\beta}$ ) such that the first nonzero element of the first nonzero row (resp. column) is equal to 1. Since  $\dim \mathfrak{g}_{\pm\beta} = 1$ ,  $X_\beta, Y_\beta$  are fixed in this way uniquely. By a straightforward calculation, one can show that

$$\{H_i, X_\beta, Y_\beta \mid i = 1, \dots, 4, \beta \in \Phi^+\}$$

form a Chevalley basis, and the element  $\rho$  is given by

$$\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} [X_\beta, Y_\beta].$$

The element  $I_+$  now reads

$$I_+ = 16 X_1 + 22 X_2 + 30 X_3 + 42 X_4.$$

We fix a basis  $\{\gamma_i\}_{i=1}^4$  of  $V = \text{Ker ad } I_+$  as follows:

$$\begin{aligned} \gamma_1 &= X_{0001} + \frac{5}{7} X_{0010} + \frac{11}{21} X_{0100} + \frac{8X_{1000}}{21}, \quad \gamma_4 = X_{2243}, \\ \gamma_2 &= X_{0122} - \frac{8}{21} X_{1121} + \frac{128}{231} X_{2021}, \quad \gamma_3 = X_{2122} + \frac{15}{8} X_{1132}. \end{aligned}$$

By using the formulae given at the beginning of the present section, we can compute the reduced Poisson brackets w.r.t. the above basis. To present the result, we introduce the following flat coordinates

$$\begin{aligned} t_1 &= u_4 - \frac{762841u_1^6}{49009212} - \frac{129973u_2u_1^3}{259308} - \frac{2783u_3u_1^2}{3528} - \frac{56741u_2^2}{142296}, \\ t_2 &= \frac{1781u_1^4}{64827} + \frac{34u_2u_1}{231} + u_3, \quad t_3 = \frac{4199u_1^3}{63504} + \frac{4199u_2}{3696}, \quad t_4 = -\frac{13u_1}{42}. \end{aligned}$$

Then the two metrics given by the coefficients of the leading terms of the reduced Poisson brackets correspond to the flat pencil of metric of the Frobenius manifolds with potential

$$\begin{aligned} F &= \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + \frac{20736 t_4^{13}}{143} + \frac{82944 t_3^2 t_4^7}{2527} + \frac{1083}{20} t_2^2 t_4^5 \\ &\quad + \frac{288}{19} t_2 t_3^2 t_4^3 + \frac{27648 t_3^4 t_4}{130321} + \frac{6859 t_2^3 t_4}{1152}, \end{aligned}$$

its Euler vector field is

$$E = \sum_{i=1}^4 E^i \frac{\partial}{\partial t_i} = t_1 \frac{\partial}{\partial t_1} + \frac{2t_2}{3} \frac{\partial}{\partial t_2} + \frac{t_3}{2} \frac{\partial}{\partial t_3} + \frac{t_4}{6} \frac{\partial}{\partial t_4}.$$

In the coordinates  $t_i$ , the first metric  $g_1^{ij}$  given by the coefficients of the leading terms of the first Poisson structure has the standard expression [16]

$$(g_1^{ij}) = (\eta_{ij})^{-1}, \quad \eta_{ij} = \partial_{t_i} \partial_{t_j} \text{Lie}_e F, \quad (8.29)$$

where the unity vector field  $e$  is given by

$$e = \frac{\partial}{\partial t_1}. \quad (8.30)$$

The second metric  $g_2^{ij}$  satisfies the formula

$$g_2^{ij}(t) = \sum_{m=1}^n E^m c_m^{ij}(t), \text{ with } c_m^{ij}(t) = g_1^{ik} g_1^{jl} \partial_{t_m} \partial_{t_k} \partial_{t_l} F. \quad (8.31)$$

Here  $n = 4$ .

The coefficients  $A_{1,0;a}^{ij}$  ( $a = 1, 2$ ) of the reduced Poisson brackets are equal to zero. The coefficients  $A_{2,0;2}^{ij}$  read

$$\begin{aligned} A_{2,0,2}^{11} &= 238464 t_4^{10} - \frac{79854336 t_3 t_4^7}{4693} + \frac{362769128 t_2 t_4^6}{37349} + \frac{82248768000 t_3^2 t_4^4}{13482989} \\ &\quad + \frac{65740256 t_1 t_4^4}{371293} - \frac{286440}{247} t_2 t_3 t_4^3 + \frac{6443534125 t_2^2 t_4^2}{2689128} - \frac{53236224 t_3^3 t_4}{1694173} \\ &\quad - \frac{4015872 t_1 t_3 t_4}{54587} + \frac{1656}{19} t_2 t_3^2 + \frac{443 t_1 t_2}{26}, \\ A_{2,0,2}^{12} &= -\frac{15818112 t_4^8}{4693} + \frac{42634554624 t_3 t_4^5}{13482989} - \frac{6453151372 t_2 t_4^4}{21163701} - \frac{51777792 t_3^2 t_4^2}{1694173} \\ &\quad - \frac{28255104 t_1 t_4^2}{709631} + \frac{7349328 t_2 t_3 t_4}{54587} + \frac{153 t_2^2}{13}, \\ A_{2,0,2}^{13} &= \frac{204693422 t_4^7}{37349} - \frac{5205718984 t_3 t_4^4}{7054567} + \frac{9722937545 t_2 t_4^3}{5378256} + \frac{3133152 t_3^2 t_4}{54587} \\ &\quad + \frac{79 t_1 t_4}{4} - \frac{3611 t_2 t_3}{312}, \\ A_{2,0,2}^{14} &= \frac{16435064 t_4^5}{1113879} + \frac{14507020 t_3 t_4^2}{709631} - \frac{2783 t_2 t_4}{312}, \\ A_{2,0,2}^{22} &= \frac{13824 t_4^6}{19} - \frac{3170304 t_3 t_4^3}{89167} + \frac{4883336 t_2 t_4^2}{125229} + \frac{13824 t_3^2}{6859} + \frac{2400 t_1}{4693}, \\ A_{2,0,2}^{23} &= -\frac{2508 t_4^5}{13} + \frac{197596 t_3 t_4^2}{2197} + \frac{817 t_2 t_4}{312}, \\ A_{2,0,2}^{24} &= \frac{39412 t_4^3}{6591} - \frac{56 t_3}{247}, \quad A_{2,0,2}^{34} = -\frac{2261 t_4^2}{624}, \quad A_{2,0,2}^{44} = \frac{13}{24}, \end{aligned}$$

and the coefficients  $A_{2,0;1}^{ij}$  is given by

$$A_{2,0;1}^{ij}(t) = \frac{\partial}{\partial t_1} A_{2,0;2}^{ij}(t). \quad (8.32)$$

Now we begin to compute the central invariants. We first find the canonical coordinates from the characteristic equation  $\det(g_2^{ij} - \lambda g_1^{ij}) = 0$ . The roots can be represented in the form

$$\begin{aligned} \lambda_{\mu_1, \mu_2} &= \left( t_1 + \frac{288}{19} t_3 t_4^3 \right) + \mu_1 \left( \frac{57}{2} t_2 t_4^2 + \frac{288}{361} t_3^2 \right) \\ &\quad + \mu_2 \frac{(361 t_2 + \mu_1 576 t_3 t_4 + 2736 t_4^4)^{\frac{3}{2}}}{228 \sqrt{57}}, \end{aligned}$$

where  $\mu_1, \mu_2 = \pm 1$ . We number them in the way that

$$\lambda_1 = \lambda_{++}, \lambda_2 = \lambda_{+-}, \lambda_3 = \lambda_{-+}, \lambda_4 = \lambda_{--}.$$

We then compute the metrics  $g_1, g_2$  and the functions  $A_{2,0;1}, A_{2,0;1}$  in the canonical coordinates. After a straightforward computation, we obtain the central invariants from the formula (2.12), they read

$$\{c_1, c_2, c_3, c_4\} = \left\{\frac{1}{24}, \frac{1}{24}, \frac{1}{12}, \frac{1}{12}\right\},$$

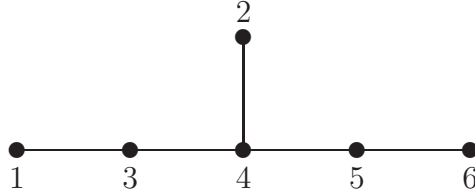
which proves the lemma for the  $F_4$  case.

### The $E_6$ case

The proof of the lemma for the simple Lie algebras of  $E$  types are similar to that of the  $F_4$  case. We take  $E_6$  for example. It has a 27-dimensional matrix realization, the Weyl generators are realized as

$$\begin{aligned} X_1 &= e_{6,7} + e_{8,9} + e_{10,11} + e_{12,14} + e_{15,17} + e_{26,27}, \\ X_2 &= e_{4,5} + e_{6,8} + e_{7,9} - e_{18,20} - e_{21,22} - e_{23,24}, \\ X_3 &= e_{4,6} + e_{5,8} + e_{11,13} + e_{14,16} + e_{17,19} + e_{25,26}, \\ X_4 &= e_{3,4} - e_{8,10} - e_{9,11} - e_{16,18} - e_{19,21} + e_{24,25}, \\ X_5 &= e_{2,3} - e_{10,12} - e_{11,14} - e_{13,16} + e_{21,23} + e_{22,24}, \\ X_6 &= e_{1,2} + e_{12,15} + e_{14,17} + e_{16,19} + e_{18,21} + e_{20,22}, \end{aligned}$$

and  $Y_i = X_i^T$ ,  $i = 1, \dots, 6$ . The Dynkin diagram for these generators are given by



The normalized Killing form is

$$\langle A, B \rangle_{\mathfrak{g}} = \frac{1}{6} \text{tr}(A B).$$

The Chevalley basis is defined in the same way. The element  $I_+$  reads

$$I_+ = 16 X_1 + 22 X_2 + 30 X_3 + 42 X_4 + 30 X_5 + 16 X_6.$$



The basis  $\{\gamma_i\}_{i=1}^6$  of  $V = \text{Ker ad } I_+$  is chosen as

$$\begin{aligned}\gamma_1 &= X_1 + \frac{11}{8}X_2 + \frac{15}{8}X_3 + \frac{21}{8}X_4 + \frac{15}{8}X_5 + X_6, \\ \gamma_2 &= X_{001111} - \frac{11}{15}X_{010111} + X_{101110} + \frac{11}{15}X_{111100}, \\ \gamma_3 &= X_{011111} - \frac{21}{8}X_{011210} - \frac{16}{11}X_{101111} - X_{111110}, \\ \gamma_4 &= X_{011221} + \frac{8}{15}X_{111211} + X_{112210}, \\ \gamma_5 &= X_{111221} + X_{112211}, \quad \gamma_6 = X_{122321}.\end{aligned}$$

The flat coordinates have the expressions

$$\begin{aligned}t_1 &= \frac{5339887u_1^6}{84934656} + \frac{129973u_3u_1^3}{442368} - \frac{2783u_4u_1^2}{77760} \\ &\quad + \frac{1679u_2^2u_1}{24300} + \frac{56741u_3^2}{1672704} + \frac{u_6}{81}, \\ t_2 &= \frac{4}{15}u_2u_1^2 + \frac{2u_5}{27}, \quad t_3 = \frac{33839u_1^4}{147456} + \frac{2261u_3u_1}{12672} - \frac{38u_4}{405}, \\ t_4 &= \frac{1547u_1^3}{9216} + \frac{221u_3}{528}, \quad t_5 = \frac{52u_2}{135}, \quad t_6 = \frac{13u_1}{8}.\end{aligned}$$

The potential of the corresponding Frobenius manifold is given by

$$\begin{aligned}F &= -\frac{3^8}{2} \left( \frac{1}{2}t_1^2t_6 + t_1t_2t_5 + t_1t_3t_4 + \frac{t_6^{13}}{185328} + \frac{1}{576}t_5^2t_6^8 + \frac{1}{252}t_4^2t_6^7 \right. \\ &\quad + \frac{1}{60}t_3^2t_6^5 + \frac{1}{24}t_4t_5^2t_6^5 + \frac{1}{24}t_2^2t_6^4 + \frac{1}{24}t_3t_5^2t_6^4 + \frac{1}{24}t_5^4t_6^3 + \frac{1}{6}t_3t_4^2t_6^3 \\ &\quad + \frac{1}{6}t_2t_4t_5t_6^3 + \frac{1}{4}t_4^2t_5^2t_6^2 + \frac{1}{2}t_2t_3t_5t_6^2 + \frac{1}{12}t_4^4t_6 + \frac{1}{6}t_3^3t_6 + \frac{1}{6}t_2t_5^3t_6 \\ &\quad \left. + \frac{1}{2}t_3t_4t_5^2t_6 + \frac{1}{2}t_2^2t_4t_6 + \frac{1}{12}t_4t_5^4 + \frac{1}{4}t_3^2t_5^2 + \frac{1}{2}t_2^2t_3 + \frac{1}{2}t_2t_4^2t_5 \right). \quad (8.33)\end{aligned}$$

Note that the function  $-\frac{2}{3^8}F(t)$  was obtained as polynomial solutions of the WDVV equations associated to the root systems of type  $E_6$  by Di Francesco *et al* in [10]. Polynomial solutions to the WDVV equations associated to the root systems of type  $E_7$  and  $E_8$  are also computed in [10].

The Euler vector field and the unity vector field have the forms

$$\begin{aligned}
E &= \sum_{k=1}^6 E^k \frac{\partial}{\partial t_k} \\
&= t_1 \frac{\partial}{\partial t_1} + \frac{3}{4} t_2 \frac{\partial}{\partial t_2} + \frac{2}{3} t_3 \frac{\partial}{\partial t_3} + \frac{1}{2} t_4 \frac{\partial}{\partial t_4} + \frac{5}{12} t_5 \frac{\partial}{\partial t_5} + \frac{1}{6} t_6 \frac{\partial}{\partial t_6}, \quad (8.34) \\
e &= \frac{1}{81} \frac{\partial}{\partial t_1}. \quad (8.35)
\end{aligned}$$

The two flat metrics  $g_1, g_2$  are expressed by the formulae given in (8.29), (8.31). We will not write down the explicit expression of the functions  $A_{2,0;1}^{ij}, A_{2,0;2}^{ij}$ , since in this case they are quite long. As a consequence of this fact, the computation of the central invariants by using the formula (8.27) becomes rather tedious. To avoid this complexity, we employ an alternative way to prove the result that the central invariants of the Drinfeld - Sokolov bihamiltonian structure related to the  $E_6$  (also for  $E_7, E_8$ ) type simple Lie algebra are equal to  $\frac{1}{24}$ .

Our approach is to establish, through an appropriate Miura type transformation, a relationship of the present bihamiltonian structure to the one defined by a semisimple Frobenius manifold via the formulae of Theorem 1 and Theorem 2 of [18]. Then the needed result follows if we can prove that the central invariants of the bihamiltonian structure given by Theorem 1 and Theorem 2 of [18] are equal to  $\frac{1}{24}$ . This fact can be proved by using properties of a semisimple Frobenius manifold. In fact, by using the formulae (3.9), (3.14), (3.15), (5.24) of [18] we can express the functions  $f^i, Q_1^{ii}, Q_2^{ii}, P_1^{ki}, P_2^{ki}$  that appear in (2.12) as follows:

$$\begin{aligned}
f^i &= \frac{1}{\psi_{i1}^2}, \quad P_1^{ki} = P_2^{ki} = 0, \\
Q_1^{ii} &= \frac{1}{12} \sum_{j=1}^n \left( \frac{\gamma_{ij}}{\psi_{i1}^3 \psi_{j1}} + \frac{\gamma_{ij} \psi_{j1}}{\psi_{i1}^5} \right), \\
Q_2^{ii} &= \frac{1}{24} \left[ \frac{1}{3\psi_{i1}^4} + 2 \sum_{j=1}^n \left( \frac{\lambda_i \gamma_{ij}}{\psi_{i1}^3 \psi_{j1}} + \frac{\lambda_i \gamma_{ij} \psi_{j1}}{\psi_{i1}^5} \right) \right].
\end{aligned}$$

Here  $n$  is the dimension of the semisimple Frobenius manifold,  $\lambda_1, \dots, \lambda_n$  are its canonical coordinates, the functions  $\gamma_{ij}$  are the rotation coefficients of the flat metric of the Frobenius manifold, and the functions  $\psi_{i1}$  are defined by (4.5) of [18]. Note that in [18] we denote the canonical coordinates by  $u_1, \dots, u_n$ . By plugging the above expressions into the formula (2.12) we immediately obtain the result  $c_i = \frac{1}{24}, i = 1, \dots, n$ .

Now let us assume that the needed Miura type transformation has the form

$$\tilde{t}_i = t_i - \epsilon^2 \left( \sum_m K_m^i t_{m,xx} + \sum_{k,l} M_{kl}^i t_{k,x} t_{l,x} \right), \quad i = 1, \dots, 6,$$

A straightforward computation shows that there is a unique choice of the  $(1, 1)$  tensor  $K_j^i$  with the following nonzero components:

$$\begin{aligned} K_3^1 &= \frac{92 t_6}{247}, \quad K_4^1 = \frac{1172287 t_6^2}{2016846}, \quad K_5^1 = \frac{3197 t_5}{4056}, \quad K_6^4 = \frac{17 t_6}{26}, \\ K_5^2 &= \frac{460 t_6}{507}, \quad K_6^2 = \frac{502 t_5}{507}, \quad K_4^3 = \frac{19}{39}, \quad K_6^3 = \frac{47120 t_6^2}{59319}, \\ K_6^1 &= \frac{2054383 t_6^4}{60149466} + \frac{7521 t_4 t_6}{5746} + \frac{115 t_3}{247}. \end{aligned}$$

such that in the new coordinates, the coefficients of  $\epsilon^2 \delta'''(x-y)$  of our reduced bihamiltonian structure can be expressed, in terms of the potential  $F(\tilde{t}) = F(t)|_{t \rightarrow \tilde{t}}$  given in (8.33), by the following formulae of Theorem 1 and Theorem 2 of [18]:

$$A_{2,0,1}^{ij}(\tilde{t}) = \frac{1}{12} \partial_{t_k} (g_1^{kl} c_l^{ij}), \quad (8.36)$$

$$A_{2,0,2}^{ij}(\tilde{t}) = \frac{1}{12} \left( \partial_{t_k} (g_2^{kl} c_l^{ij}) + \frac{1}{2} c_l^{kl} c_k^{ij} \right), \quad (8.37)$$

where  $g_1^{ij}(\tilde{t})$ ,  $g_2^{ij}(\tilde{t})$ ,  $c_k^{ij}(\tilde{t})$  are defined as in (8.29), (8.31) by using the function  $F(t)$  and the vector fields (8.34), (8.35), and then replacing  $t$  by  $\tilde{t}$ .

Since in the present case the central invariants are determined by the coefficients of  $\epsilon^2 \delta'''(x-y)$  of the bihamiltonian structure, the above Miura type transformation (with arbitrary chosen functions  $M_{kl}^i$ ) already establishes the fact that all the central invariants of the bihamiltonian structure that we are considering are equal to  $\frac{1}{24}$ .

For the simple Lie algebra of type  $E_7$ ,  $E_8$ , we give the relevant data in the Appendix A and B, the notations are in agreement with that of the above  $E_6$  case. We thus complete the proof of the lemma.  $\square$

## 9 Conclusion

In this paper, we compute the central invariants of the bihamiltonian structures of Drinfeld - Sokolov reduction related to the affine Kac-Moody algebras of type  $A_n^{(1)}$ ,  $B_n^{(1)}$ ,  $C_n^{(1)}$ ,  $D_n^{(1)}$ ,  $G_2^{(1)}$ ,  $F_4^{(1)}$ ,  $E_{6,7,8}^{(1)}$  with the standard gradation which is given by the vertex  $c_0$  in the (extended) Dynkin diagram.

For the standard gradations defined by another vertex, Drinfeld and Sokolov didn't give the bihamiltonian structures. We point out that the generalized KdV equations for other standard gradations do possess bihamiltonian structures of the form (7.7), but in general these bihamiltonian structures have infinite many terms. This is because all these equations are related to the generalized mKdV equations through a Miura type transformation, while these transformations are invertible in the formal power series sense. This fact has an immediate corollary that the central invariants of these bihamiltonian structures are the same with the ones we have computed.

The generalized KdV equations related to the *twisted* affine Lie algebras seem not to possess a bihamiltonian structure. We give here a counterexample.

Let us consider the generalized KdV equation related to  $A_2^{(2)}$  equipped with the standard gradation defined by the vertex  $c_0$ . The simplest integrable equation reads [13]

$$u_t = 5u^2u_x + 5\epsilon^2(u_xu_{xx} + uu_{xxx}) + \epsilon^4u_{xxxx}. \quad (9.1)$$

**Proposition 9.1** *The equation (9.1) possesses only one local Hamiltonian structure found in [13]*

$$u_t = \{u(x), H\}, \quad H = \int (u^3 - 3u_x^2) dx \quad (9.2)$$

$$\{u(x), u(y)\} = 2u(x)\delta'(x-y) + u_x\delta(x-y) + \frac{\epsilon^2}{2}\delta'''(x-y).$$

The proof was obtained in [38] following the scheme of [40]. Let us give here the sketch of the proof. First, we construct the so-called *quasitriviality*

transformation  $v \mapsto u$ ,

$$\begin{aligned}
u = & v + \frac{\epsilon^2}{4} \partial_x^2 (\log v_1 - \log v) + \\
& + \epsilon^4 \partial_x^2 \left( \frac{32 v_2}{v^2} - \frac{27 v_1^2}{v^3} - \frac{12 v_3}{v v_1} + \frac{5 v_4}{v_1^2} + \frac{11 v_2^2}{v v_1^2} - \frac{21 v_3 v_2}{v_1^3} + \frac{16 v_2^3}{v_1^4} \right) \\
& + \epsilon^6 \partial_x^2 \left[ \frac{7533 v_1^4}{4480 v^6} - \frac{43081 v_2 v_1^2}{13440 v^5} + \frac{2063 v_3 v_1}{1920 v^4} + \frac{2077 v_2^2}{3360 v^4} - \frac{619 v_4}{2240 v^3} \right. \\
& + \frac{1}{v_1} \left( \frac{239 v_2 v_3}{1344 v^3} + \frac{41 v_5}{672 v^2} \right) - \frac{1}{v_1^2} \left( \frac{157 v_2^3}{1920 v^3} + \frac{1549 v_4 v_2}{6720 v^2} + \frac{13 v_3^2}{80 v^2} + \frac{7 v_6}{640 v} \right) \\
& + \frac{1}{v_1^3} \left( \frac{8753 v_3 v_2^2}{13440 v^2} + \frac{383 v_5 v_2}{4480 v} + \frac{689 v_3 v_4}{4480 v} + \frac{v_7}{384} \right) \\
& - \frac{1}{v_1^4} \left( \frac{4303 v_2^4}{13440 v^2} + \frac{185 v_4 v_2^2}{448 v} + \frac{2607 v_3^2 v_2}{4480 v} + \frac{21 v_2 v_6}{640} + \frac{103 v_4^2}{2240} + \frac{159 v_3 v_5}{2240} \right) \\
& + \frac{1}{v_1^5} \left( \frac{9343 v_3 v_2^3}{6720 v} + \frac{1059 v_5 v_2^2}{4480} + \frac{3819 v_3 v_4 v_2}{4480} + \frac{177 v_3^3}{896} \right) \\
& \left. - \frac{1}{v_1^6} \left( \frac{131 v_2^5}{210 v} + \frac{83}{70} v_4 v_2^3 + \frac{2241}{896} v_3^2 v_2^2 \right) + \frac{59 v_2^4 v_3}{14 v_1^7} - \frac{5 v_2^6}{3 v_1^8} \right] + O(\epsilon^8) \quad (9.3)
\end{aligned}$$

transforming any monotone solution of the dispersionless equation

$$v_t = 5v^2 v_x$$

to a solution of (9.1). In this long formula we denote the jet coordinates by  $v_1 = v_x$ ,  $v_2 = v_{xx}$  etc. According to [38, 40], any local Hamiltonian structure of (9.1) with coefficients depending *polynomially* on the jet coordinates  $u_x$ ,  $u_{xx}$ ,  $\dots$  must be obtained from some dispersionless Hamiltonian structure of the form

$$\{v(x), v(y)\} = \varphi(v(x)) \delta'(x - y) + \frac{1}{2} \varphi'(v(x)) v_x(x) \delta(x - y)$$

by applying the quasitriviality transformation (9.3). The unknown function  $\varphi(v)$  has to be chosen in such a way to ensure cancellation of all the jet dependent denominators in the transformed bracket

$$\{u(x), u(y)\} = \varphi(u) \delta'(x - y) + \frac{1}{2} \varphi'(u) u_x \delta(x - y) + \epsilon^2 Z_2 + \epsilon^4 Z_4 + \epsilon^6 Z_6 + \dots \quad (9.4)$$

Here  $Z_2$  is a polynomial for any  $\varphi(v)$ , while  $Z_4$  contains the following term

$$-\frac{3}{160} \frac{u_{xx}^4}{u^2 u_x^4} [3 u^2 \varphi''(u) - 2 u \varphi'(u) + 2 \varphi(u)] \delta'(x - y).$$

So, to ensure  $Z_4$  is a polynomial, we must have

$$\varphi(u) = c_1 u + c_2 u^{\frac{2}{3}}$$

for some constants  $c_1$  and  $c_2$ . Next,  $Z_6$  contains the following term

$$\frac{5 c_2 u_{xx}^5}{432 u^{10/3} u_x^4} \delta'(x - y),$$

which implies  $c_2 = 0$ . So we have  $\varphi(u) = c_1 u$ , by taking  $c_1 = 2$ , we obtain the Hamiltonian structure (9.2). The Proposition is proved.

In a similar way we have analyzed another example of an integrable scalar equation associated with  $A_2^{(2)}$ . It would be interesting to prove in general that the Drinfeld - Sokolov hierarchies associated with twisted Kac - Moody Lie algebras never admit a local bihamiltonian structure.

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## A Data for $E_7$

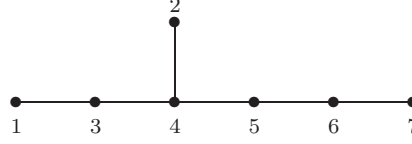
The Weyl basis

$$\begin{aligned} X_1 &= e_{7,8} + e_{9,10} + e_{11,12} + e_{13,15} + e_{16,18} + e_{19,22} - e_{35,38} - e_{39,41} - e_{42,44} - e_{45,46} - e_{47,48} - e_{49,50}, \\ X_2 &= e_{5,6} + e_{7,9} + e_{8,10} - e_{20,23} - e_{24,26} - e_{27,29} - e_{28,30} - e_{31,33} - e_{34,37} + e_{47,49} + e_{48,50} + e_{51,52}, \\ X_3 &= e_{5,7} + e_{6,9} + e_{12,14} + e_{15,17} + e_{18,21} + e_{22,25} - e_{32,35} - e_{36,39} - e_{40,42} - e_{43,45} + e_{48,51} + e_{50,52}, \\ X_4 &= e_{4,5} - e_{9,11} - e_{10,12} - e_{17,20} - e_{21,24} - e_{25,28} - e_{29,32} - e_{33,36} - e_{37,40} + e_{45,47} + e_{46,48} - e_{52,53}, \\ X_5 &= e_{3,4} - e_{11,13} - e_{12,15} - e_{14,17} - e_{24,27} - e_{26,29} - e_{28,31} - e_{30,33} + e_{40,43} + e_{42,45} + e_{44,46} - e_{53,54}, \\ X_6 &= e_{2,3} - e_{13,16} - e_{15,18} - e_{17,21} - e_{20,24} - e_{23,26} + e_{31,34} + e_{33,37} + e_{36,40} + e_{39,42} + e_{41,44} - e_{54,55}, \\ X_7 &= e_{1,2} + e_{16,19} + e_{18,22} + e_{21,25} + e_{24,28} + e_{26,30} + e_{27,31} + e_{29,33} + e_{32,36} + e_{35,39} + e_{38,41} + e_{55,56}, \\ Y_i &= X_i^T, \quad H_i = [X_i, Y_i], \quad i = 1, \dots, 7. \end{aligned}$$

The invariant bilinear form

$$\langle A, B \rangle_{\mathfrak{g}} = \frac{1}{12} \text{tr}(A B).$$

The Dynkin diagram



The basis of  $V$

$$\begin{aligned}
\gamma_1 &= X_7 + \frac{52}{27}X_6 + \frac{25}{9}X_5 + \frac{32}{9}X_4 + \frac{22}{9}X_3 + \frac{49}{27}X_2 + \frac{34}{27}X_1 = \frac{1}{27}I_+, \\
\gamma_2 &= X_{00111111} - \frac{49}{44}X_{01011111} - \frac{49}{54}X_{01111110} + \frac{196}{117}X_{0112100} + \frac{34}{27}X_{10111110} + \frac{833}{1404}X_{11111100}, \\
\gamma_3 &= X_{01121111} - \frac{25}{9}X_{0112210} + \frac{17}{24}X_{11111111} - \frac{34}{27}X_{1112110} - \frac{187}{117}X_{1122100}, \\
\gamma_4 &= X_{0112221} + \frac{17}{26}X_{1112211} + \frac{187}{325}X_{1122111}, \gamma_5 = X_{1122221} - \frac{24}{13}X_{1123211} + \frac{392}{117}X_{1223210}, \\
\gamma_6 &= X_{1123321} - \frac{49}{75}X_{1223221}, \gamma_7 = X_{2234321}.
\end{aligned}$$

The flat coordinates of the first metric

$$\begin{aligned}
t_1 &= \frac{38484341090380842575u_1^9}{8105110306037952534} - \frac{636900975026373245u_2u_1^6}{58884863779069614} + \frac{36219697127845u_3u_1^5}{132177023073108} \\
&\quad - \frac{17412855149u_4u_1^4}{16788221190} + \frac{2705810236u_5u_1^3}{5036466357} + \frac{6468204279499075u_2^3u_1^3}{2566845151777764} \\
&\quad - \frac{4934129232697u_2u_3u_1^2}{2880858756204} - \frac{553112u_6u_1^2}{13286025} - \frac{2474855971u_2^3u_1}{38799444528} + \frac{270317u_2u_4u_1}{1108809} \\
&\quad - \frac{624221464415u_2^3}{7944932410272} - \frac{329273u_3u_4}{9240075} - \frac{433415u_2u_5}{16888014} - \frac{u_7}{81}, \\
t_2 &= -\frac{320145055952981u_1^7}{68630377364883} + \frac{2127572498737u_2u_1^4}{460255540932} - \frac{3585503245u_3u_1^3}{1549681956} + \frac{160921u_4u_1^2}{328050} \\
&\quad - \frac{19277059375u_2^2u_1}{20062960632} - \frac{1798u_5u_1}{19683} + \frac{11494397u_2u_3}{67552056} + \frac{124u_6}{2025}, \\
t_3 &= \frac{333872584054u_1^6}{847288609443} + \frac{8686821907u_2u_1^3}{3357644238} + \frac{24931445u_3u_1^2}{165809592} + \frac{7337u_4u_1}{52650} - \frac{44605625u_2^2}{495381744} - \frac{725u_5}{3159}, \\
t_4 &= \frac{162925u_1^5}{59049} - \frac{149891u_2u_1^2}{69498} + \frac{665u_3u_1}{1404} - \frac{161u_4}{650}, \\
t_5 &= \frac{92912470u_1^4}{43046721} - \frac{382375u_2u_1}{938223} + \frac{11305u_3}{12636}, \\
t_6 &= -\frac{2185u_2}{1782} + \frac{439622u_1^3}{531441}, \quad t_7 = -\frac{133u_1}{27}.
\end{aligned}$$

The Euler and the unity vector fields

$$\begin{aligned}
E &= t_1 \frac{\partial}{\partial t_1} + \frac{7}{9}t_2 \frac{\partial}{\partial t_2} + \frac{2}{3}t_3 \frac{\partial}{\partial t_3} + \frac{5}{9}t_4 \frac{\partial}{\partial t_4} + \frac{4}{9}t_5 \frac{\partial}{\partial t_5} + \frac{1}{3}t_6 \frac{\partial}{\partial t_6} + \frac{1}{9}t_7 \frac{\partial}{\partial t_7}, \\
e &= -\frac{1}{81} \frac{\partial}{\partial t_1}.
\end{aligned}$$

The potential of the Frobenius manifold is given by  $F(t) = \frac{37}{2}\tilde{F}$ , where  $\tilde{F}$  is obtained from the  $E_7$  free energy of [10](given in Appendix D) by the following substitution:

$$t_0 \mapsto t_1, \quad t_4 \mapsto t_2, \quad t_6 \mapsto t_3, \quad t_8 \mapsto t_4, \quad t_{10} \mapsto t_5, \quad t_{12} \mapsto t_6, \quad t_{16} \mapsto t_7.$$

The components of the tensor  $K$ :

$$\begin{aligned}
K_2^1 &= \frac{1265t_7}{5301}, K_3^1 = -\frac{572296t_7^2}{2355525}, K_4^1 = \frac{158794t_6}{211071} - \frac{3974393683t_7^3}{46960314975}, \\
K_5^1 &= -\frac{44538734101583t_7^4}{4034736342022050} - \frac{23803046371t_6t_7}{45336995445} - \frac{1049501t_5}{9665775}, \\
K_6^1 &= -\frac{135026437662337483t_7^5}{22921337159027266050} + \frac{4216764458944t_6t_7^2}{20156828174847} - \frac{3478865269t_5t_7}{6476713635} + \frac{974770t_4}{1175967}, \\
K_7^1 &= \frac{11916123469344407708t_7^7}{33997824196158072507075} - \frac{833538047791538069t_6t_7^4}{22921337159027266050} - \frac{12187906171012t_5t_7^3}{288195453001575} \\
&\quad - \frac{3127213133t_4t_7^2}{9690223725} + \frac{18075424576t_6^2t_7}{73834535439} - \frac{2899844t_3t_7}{4616829} + \frac{1883t_2}{5301} - \frac{44290188771t_5t_6}{75561659075}, \\
K_3^2 &= \frac{31}{75}, K_4^2 = \frac{4029628t_7}{4817925}, K_5^2 = -\frac{19829032882t_7^2}{68991080025}, K_4^3 = -\frac{29}{42}, K_5^4 = \frac{27}{34}, \\
K_6^2 &= \frac{33964731481462t_7^3}{460101512686725} - \frac{33203728t_6}{22343373}, K_5^3 = -\frac{411452t_7}{2706417}, K_6^3 = -\frac{213705872t_7^2}{668484999}, \\
K_7^2 &= -\frac{27908652530225533t_7^5}{3488029567678062225} + \frac{4267079088702t_6t_7^2}{153367170895575} - \frac{387454306t_5t_7}{579756975} + \frac{236468t_4}{229425}, \\
K_7^3 &= -\frac{6028481916881t_7^4}{456100629967710} - \frac{184848088t_6t_7}{257109615} - \frac{21373t_5}{159201}, K_6^4 = -\frac{4518t_7}{4199}, \\
K_7^4 &= \frac{7249042t_7^3}{53054365} - \frac{1926t_6}{1615}, K_6^5 = \frac{25}{39}, K_7^5 = -\frac{939370t_7^2}{2069613}, K_6^6 = \frac{4922t_7}{8379}.
\end{aligned}$$

## B Data for $E_8$

The Weyl basis

$$\begin{aligned}
X_1 &= e_{8,9} + e_{10,11} + e_{12,13} + e_{14,16} + e_{17,19} + e_{20,23} + e_{24,28} - e_{42,47} - e_{48,53} - e_{54,59} - e_{56,61} \\
&\quad - e_{60,65} - e_{62,67} - e_{66,71} - e_{68,73} - e_{69,75} - e_{72,78} - e_{74,80} - e_{76,82} - e_{81,86} - e_{83,88} - e_{84,90} \\
&\quad - e_{89,94} - e_{91,96} - e_{97,102} - e_{98,104} - e_{105,110} - e_{112,118} - e_{120,127} + e_{135,143} + e_{142,149} \\
&\quad + e_{148,155} + e_{150,157} + e_{154,161} + e_{156,163} + e_{160,167} + e_{162,169} + e_{164,171} + e_{166,172} + e_{168,174} \\
&\quad + e_{170,176} + e_{173,179} + e_{175,181} + e_{177,184} + e_{180,185} + e_{182,188} + e_{186,192} + e_{189,195} + e_{193,198} \\
&\quad + e_{199,205} - e_{224,228} - e_{225,229} - e_{230,232} - e_{233,235} - e_{236,237} - e_{238,239} - e_{240,241} + 2e_{127,136} \\
&\quad + \frac{1}{2}(e_{122,136} - e_{121,136} - e_{123,136} + e_{124,136} - e_{125,136} - 3e_{126,136} - e_{128,136}),
\end{aligned}$$

$$\begin{aligned}
X_2 &= e_{6,7} + e_{8,10} + e_{9,11} - e_{21,25} - e_{26,29} - e_{30,33} - e_{31,34} - e_{35,38} - e_{36,40} - e_{39,44} - e_{41,45} \\
&\quad - e_{46,51} - e_{52,58} + e_{66,72} + e_{71,78} + e_{74,81} + e_{79,85} + e_{80,86} + e_{83,89} + e_{87,93} + e_{88,94} + e_{91,97} \\
&\quad + e_{95,101} + e_{96,102} + e_{98,105} + e_{103,109} + e_{104,110} + e_{111,117} + e_{119,125} + e_{119,126} + e_{119,127} \\
&\quad + e_{125,134} + e_{126,134} + e_{133,141} + e_{140,147} + e_{142,150} + e_{146,153} + e_{148,156} + e_{149,157} + e_{152,159} \\
&\quad + e_{154,162} + e_{155,163} + e_{158,165} + e_{160,168} + e_{161,169} + e_{166,173} + e_{167,174} + e_{172,179} - e_{196,202} \\
&\quad - e_{200,206} - e_{203,208} - e_{207,211} - e_{209,213} - e_{212,216} - e_{214,217} - e_{215,218} - e_{219,222} - e_{223,227} \\
&\quad + e_{238,240} + e_{239,241} + e_{242,243},
\end{aligned}$$



$$\begin{aligned}
X_3 = & e_{6,8} + e_{7,10} + e_{13,15} + e_{16,18} + e_{19,22} + e_{23,27} + e_{28,32} - e_{37,42} - e_{43,48} - e_{49,54} - e_{50,56} \\
& - e_{55,60} - e_{57,62} - e_{63,68} - e_{64,69} - e_{70,76} - e_{71,79} - e_{77,84} - e_{78,85} - e_{80,87} - e_{86,93} - e_{88,95} \\
& - e_{94,101} - e_{96,103} - e_{102,109} - e_{104,111} - e_{110,117} + e_{112,120} - e_{118,125} + e_{118,126} + e_{118,127} \\
& + e_{125,135} - e_{127,135} + e_{133,142} - e_{136,143} + e_{140,148} + e_{141,150} + e_{146,154} + e_{147,156} + e_{152,160} \\
& + e_{153,162} + e_{158,166} + e_{159,168} + e_{165,173} + e_{171,178} + e_{176,183} + e_{181,187} + e_{184,190} + e_{185,191} \\
& + e_{188,194} + e_{192,197} + e_{195,201} + e_{198,204} + e_{205,210} - e_{220,224} - e_{221,225} - e_{226,230} - e_{231,233} \\
& - e_{234,236} - e_{239,242} - e_{241,243},
\end{aligned}$$

$$\begin{aligned}
X_4 = & e_{5,6} - e_{10,12} - e_{11,13} - e_{18,21} - e_{22,26} - e_{27,31} - e_{32,36} - e_{33,37} - e_{38,43} - e_{44,49} - e_{45,50} \\
& - e_{51,57} - e_{58,64} - e_{60,66} - e_{65,71} - e_{68,74} - e_{73,80} - e_{76,83} - e_{82,88} - e_{84,91} + e_{85,92} - e_{90,96} \\
& + e_{93,100} + e_{101,108} + e_{105,112} + e_{109,116} + e_{110,118} + e_{111,119} + e_{117,124} + e_{117,125} + e_{124,133} \\
& + e_{125,133} + e_{132,140} + e_{134,141} + e_{135,142} + e_{139,146} + e_{143,149} + e_{145,152} + e_{151,158} - e_{156,164} \\
& - e_{162,170} - e_{163,171} - e_{168,175} - e_{169,176} - e_{173,180} - e_{174,181} - e_{179,185} - e_{190,196} - e_{194,200} \\
& - e_{197,203} - e_{201,207} - e_{204,209} - e_{210,215} - e_{216,220} - e_{217,221} - e_{222,226} - e_{227,231} - e_{236,238} \\
& - e_{237,239} + e_{243,244},
\end{aligned}$$

$$\begin{aligned}
X_5 = & e_{4,5} - e_{12,14} - e_{13,16} - e_{15,18} - e_{26,30} - e_{29,33} - e_{31,35} - e_{34,38} - e_{36,41} - e_{40,45} - e_{49,55} \\
& - e_{54,60} - e_{57,63} - e_{59,65} - e_{62,68} - e_{64,70} - e_{67,73} - e_{69,76} - e_{75,82} + e_{91,98} + e_{92,99} + e_{96,104} \\
& + e_{97,105} + e_{100,107} + e_{102,110} + e_{103,111} + e_{108,115} + e_{109,117} + e_{116,123} + e_{116,124} + e_{123,132} \\
& + e_{124,132} + e_{131,139} + e_{133,140} + e_{138,145} + e_{141,147} + e_{142,148} + e_{144,151} + e_{149,155} + e_{150,156} \\
& + e_{157,163} - e_{170,177} - e_{175,182} - e_{176,184} - e_{180,186} - e_{181,188} - e_{183,190} - e_{185,192} - e_{187,194} \\
& - e_{191,197} - e_{207,212} - e_{209,214} - e_{211,216} - e_{213,217} - e_{215,219} - e_{218,222} - e_{231,234} - e_{233,236} \\
& - e_{235,237} + e_{244,245},
\end{aligned}$$

$$\begin{aligned}
X_6 = & e_{3,4} - e_{14,17} - e_{16,19} - e_{18,22} - e_{21,26} - e_{25,29} - e_{35,39} - e_{38,44} - e_{41,46} - e_{43,49} - e_{45,51} \\
& - e_{48,54} - e_{50,57} - e_{53,59} - e_{56,62} - e_{61,67} + e_{70,77} + e_{76,84} + e_{82,90} + e_{83,91} + e_{88,96} + e_{89,97} \\
& + e_{94,102} + e_{95,103} + e_{99,106} + e_{101,109} + e_{107,114} + e_{108,116} + e_{115,122} + e_{115,123} + e_{122,131} \\
& + e_{123,131} + e_{130,138} + e_{132,139} + e_{137,144} + e_{140,146} + e_{147,153} + e_{148,154} + e_{155,161} + e_{156,162} \\
& + e_{163,169} + e_{164,170} + e_{171,176} + e_{178,183} - e_{182,189} - e_{186,193} - e_{188,195} - e_{192,198} - e_{194,201} \\
& - e_{197,204} - e_{200,207} - e_{203,209} - e_{206,211} - e_{208,213} - e_{219,223} - e_{222,227} - e_{226,231} - e_{230,233} \\
& - e_{232,235} + e_{245,246},
\end{aligned}$$

$$\begin{aligned}
X_7 = & e_{2,3} - e_{17,20} - e_{19,23} - e_{22,27} - e_{26,31} - e_{29,34} - e_{30,35} - e_{33,38} - e_{37,43} - e_{42,48} + e_{46,52} \\
& - e_{47,53} + e_{51,58} + e_{57,64} + e_{62,69} + e_{63,70} + e_{67,75} + e_{68,76} + e_{73,82} + e_{74,83} + e_{80,88} + e_{81,89} \\
& + e_{86,94} + e_{87,95} + e_{93,101} + e_{100,108} + e_{106,113} + e_{107,115} + e_{114,121} + e_{114,122} + e_{121,130} \\
& + e_{122,130} + e_{129,137} + e_{131,138} + e_{139,145} + e_{146,152} + e_{153,159} + e_{154,160} + e_{161,167} + e_{162,168} \\
& + e_{169,174} + e_{170,175} + e_{176,181} + e_{177,182} + e_{183,187} + e_{184,188} + e_{190,194} - e_{193,199} + e_{196,200} \\
& - e_{198,205} + e_{202,206} - e_{204,210} - e_{209,215} - e_{213,218} - e_{214,219} - e_{217,222} - e_{221,226} - e_{225,230} \\
& - e_{229,232} + e_{246,247},
\end{aligned}$$

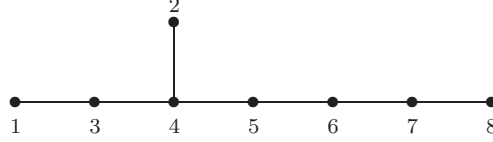
$$\begin{aligned}
X_8 = & e_{1,2} + e_{20,24} + e_{23,28} + e_{27,32} + e_{31,36} + e_{34,40} + e_{35,41} + e_{38,45} + e_{39,46} + e_{43,50} + e_{44,51} \\
& + e_{48,56} + e_{49,57} + e_{53,61} + e_{54,62} + e_{55,63} + e_{59,67} + e_{60,68} + e_{65,73} + e_{66,74} + e_{71,80} + e_{72,81} \\
& + e_{78,86} + e_{79,87} + e_{85,93} + e_{92,100} + e_{99,107} + e_{106,114} + 2e_{113,121} - e_{113,122} + e_{113,123} \\
& - e_{113,124} + e_{113,125} - e_{113,126} + e_{113,127} + e_{113,128} + e_{121,129} + e_{130,137} + e_{138,144} + e_{145,151} \\
& + e_{152,158} + e_{159,165} + e_{160,166} + e_{167,172} + e_{168,173} + e_{174,179} + e_{175,180} + e_{181,185} + e_{182,186} \\
& + e_{187,191} + e_{188,192} + e_{189,193} + e_{194,197} + e_{195,198} + e_{200,203} + e_{201,204} + e_{206,208} + e_{207,209} \\
& + e_{211,213} + e_{212,214} + e_{216,217} + e_{220,221} + e_{224,225} + e_{228,229} + e_{247,248}.
\end{aligned}$$

$$\begin{aligned}
Y_1 &= X_1^T - e_{127,120} - e_{136,127} + \frac{1}{2}(e_{121,120} - e_{122,120} + e_{123,120} - e_{124,120} + e_{125,120} + 3e_{126,120} \\
&\quad e_{128,120} + e_{136,121} - e_{136,122} + e_{136,123} - e_{136,124} + e_{136,125} + 3e_{136,126} + e_{136,128}), \\
Y_2 &= X_2^T + e_{134,127} - e_{127,119}, \quad Y_3 = X_3^T - e_{126,118} - e_{135,126}, \quad Y_i = X_i^T \quad (i = 4, 5, 6, 7), \\
Y_8 &= X_8^T - e_{121,113} + e_{122,113} - e_{123,113} + e_{124,113} - e_{125,113} + e_{126,113} - e_{127,113} - e_{128,113} \\
&\quad + e_{129,121} - e_{129,122} + e_{129,123} - e_{129,124} + e_{129,125} - e_{129,126} + e_{129,127} + e_{129,128}, \\
H_i &= [X_i, Y_i], \quad i = 1, \dots, 8.
\end{aligned}$$

The invariant bilinear form

$$\langle A, B \rangle_{\mathfrak{g}} = \frac{1}{60} \text{tr}(AB).$$

The Dynkin diagram



The basis of  $V$

$$\begin{aligned}
\gamma_1 &= X_8 + \frac{57}{29}X_7 + \frac{84}{29}X_6 + \frac{110}{29}X_5 + \frac{135}{29}X_4 + \frac{91}{29}X_3 + \frac{68}{29}X_2 + \frac{46}{29}X_1 = \frac{1}{58}I_+, \\
\gamma_2 &= X_{01111111} - \frac{135}{29}X_{01121110} + \frac{4950}{551}X_{01122100} - \frac{69}{34}X_{10111111} - \frac{92}{29}X_{11111110} \\
&\quad + \frac{2070}{551}X_{11121100} + \frac{4485}{1102}X_{11221000}, \\
\gamma_3 &= X_{01122221} + \frac{46}{57}X_{11122211} + \frac{299}{684}X_{11222111} + \frac{2093}{1653}X_{11222210} - \frac{4485}{2204}X_{11232110} \\
&\quad + \frac{25415}{10469}X_{12232100}, \\
\gamma_4 &= X_{11222221} - \frac{45}{19}X_{11232211} + \frac{4950}{551}X_{11233210} + \frac{510}{133}X_{12232111} - \frac{3060}{551}X_{12232210}, \\
\gamma_5 &= X_{12233321} - \frac{45}{28}X_{12243221} + \frac{195}{76}X_{12343211} - \frac{4485}{1102}X_{22343210}, \\
\gamma_6 &= X_{12244321} - \frac{91}{110}X_{12343321} + \frac{299}{660}X_{22343221}, \\
\gamma_7 &= X_{22454321} + \frac{68}{91}X_{23354321}, \quad \gamma_8 = X_{23465432}.
\end{aligned}$$

The flat coordinates of the first metric

$$\begin{aligned}
t_1 &= \frac{97744996564015238279820370313216u_1^{15}}{1213479667630994687008453125} - \frac{1288270064507206171704653824u_2u_1^{11}}{12314889544759321146875} \\
&\quad - \frac{2393328775550101435187456u_3u_1^9}{736464332431224703125} - \frac{2420766825182080240896u_4u_1^8}{1039574576934578125} \\
&\quad + \frac{862526917251409198851419904u_2^2u_1^7}{23648688008069326578125} - \frac{131350976991158976u_5u_1^6}{1236117213953125} \\
&\quad + \frac{1245108893989927485504u_2u_3u_1^5}{1083721620290046875} - \frac{963126559584u_6u_1^5}{121563921875} \\
&\quad + \frac{14299027732501070976u_2u_4u_1^4}{13767788279546875} - \frac{33722086688072420883336u_2^3u_1^3}{16483976224411765625} \\
&\quad + \frac{18146063436077468u_3^2u_1^3}{1440211987515625} - \frac{461307312u_7u_1^3}{1195796875} + \frac{6018680488536u_3u_4u_1^2}{630920828125} \\
&\quad + \frac{357801015222144u_2u_5u_1^2}{16370735171875} + \frac{32227564652712u_4^2u_1}{6740890953125} - \frac{1845004006277946u_2^2u_3u_1}{44434852609375} \\
&\quad + \frac{130653081u_2u_6u_1}{169468750} - \frac{178420747189383u_2^2u_4}{14647499890625} + \frac{561100797u_3u_5}{3480942500} + \frac{243u_8}{15625},
\end{aligned}$$

$$\begin{aligned}
t_2 &= \frac{3840041279133973715980288u_1^{12}}{663402718510254451875} - \frac{744229812509520647424u_2u_1^8}{100987244616501875} \\
&\quad - \frac{10589318985400591616u_3u_1^6}{58380050133271875} - \frac{53628582258067968u_4u_1^5}{247223442790625} + \frac{1500712331u_3^2}{9448272500} \\
&\quad + \frac{280559110180100677056u_2^2u_1^4}{193929132051903125} - \frac{2377142258796u_5u_1^3}{293963665625} - \frac{96129730861884u_2^3}{7114499946875} \\
&\quad - \frac{998361u_6u_1^2}{2628125} + \frac{4615114574016u_2u_4u_1}{172323528125} + \frac{383963689278654u_2u_3u_1^2}{8886970521875} - \frac{12879u_7}{284375}, \\
t_3 &= \frac{2549753401933430769664u_1^{10}}{2366478187313630625} - \frac{26925773335669824u_2u_1^6}{36655987156625} + \frac{266785048403496u_2^2u_1^2}{2427299981875} \\
&\quad - \frac{7567792181264u_3u_1^4}{243569894375} - \frac{8241329088u_4u_1^3}{442050625} - \frac{364203u_5u_1}{525625} + \frac{2292843u_2u_3}{1779730} - \frac{2709u_6}{27500}, \\
t_4 &= \frac{1522342602122771968u_1^9}{5440179740950875} - \frac{537827687557056u_2u_1^5}{1412705387375} - \frac{1371394734296u_3u_1^3}{159580275625} \\
&\quad - \frac{19495261224u_4u_1^2}{2027335625} + \frac{2320060511106u_2^2u_1}{83699999375} - \frac{1925073u_5}{4821250}, \\
t_5 &= \frac{98329903451648u_1^7}{1293740723175} - \frac{20835557808u_2u_1^3}{335958475} - \frac{1420978u_3u_1}{1308625} - \frac{887778u_4}{482125}, \quad t_8 = \frac{124u_1}{29}, \\
t_6 &= \frac{567020228224u_1^6}{14870583025} - \frac{2551308804u_2u_1^2}{196941175} - \frac{1271u_3}{1805}, \quad t_7 = \frac{22811536u_1^4}{3536405} - \frac{10323u_2}{2465}.
\end{aligned}$$

The Euler and the unity vector fields

$$\begin{aligned}
E &= t_1 \frac{\partial}{\partial t_1} + \frac{4}{5} t_2 \frac{\partial}{\partial t_2} + \frac{2}{3} t_3 \frac{\partial}{\partial t_3} + \frac{3}{5} t_4 \frac{\partial}{\partial t_4} + \frac{7}{15} t_5 \frac{\partial}{\partial t_5} + \frac{2}{5} t_6 \frac{\partial}{\partial t_6} + \frac{4}{15} t_7 \frac{\partial}{\partial t_7} + \frac{1}{15} t_8 \frac{\partial}{\partial t_8}, \\
e &= \frac{3^5}{5^6} \frac{\partial}{\partial t_1}.
\end{aligned}$$

The potential of the Frobenius manifold is given by  $F(t) = -\frac{5^{12}}{2 \cdot 3^{11}} \tilde{F}$ , where  $\tilde{F}$  is obtained from the  $E_8$  free energy of [10] (given in Appendix D) by the following substitution:

$$t_0 \mapsto t_1, \quad t_6 \mapsto t_2, \quad t_{10} \mapsto t_3, \quad t_{12} \mapsto t_4, \quad t_{16} \mapsto t_5, \quad t_{18} \mapsto t_6, \quad t_{22} \mapsto t_7, \quad t_{28} \mapsto t_8.$$

The components of the tensor  $K$ :

$$\begin{aligned}
K_2^1 &= \frac{1606747t_8^2}{1018660}, \quad K_3^1 = \frac{601725604717t_8^4}{555959642000} + \frac{18606712t_7}{8631175}, \\
K_4^1 &= \frac{23515325188849703t_8^5}{51379248583848000} + \frac{72379436165441t_7t_8}{10901267734900}, \\
K_5^1 &= \frac{68957356685371184639t_8^7}{4819885123299777312000} + \frac{398354166156373021t_7t_8^3}{82917361810316400} \\
&\quad + \frac{1042486747513t_6t_8}{287607914708} + \frac{33089674283t_5}{9730244164}, \\
K_6^1 &= \frac{89894219185919589074208393t_8^8}{740233940665443716429200000} + \frac{540313685410365273433t_7t_8^4}{372713541337372218000} \\
&\quad + \frac{431357837132874729t_6t_8^2}{61633372838501000} + \frac{56454979942439t_5t_8}{14380395735400} + \frac{33101974334668t_7^2}{8987747334625}, \\
K_7^1 &= \frac{1357568393016120771637825259t_8^{10}}{86751806948718464815666000000} + \frac{541717459607162768848213339t_7t_8^6}{408322593076744759707720000} \\
&\quad + \frac{63779850971187562357459t_6t_8^4}{35407786427050360710000} + \frac{85588038590694599881t_5t_8^3}{15754298743960116000} + \frac{4464353t_3}{1726235} \\
&\quad + \frac{6210460349164259675293t_7^2t_8^2}{728636316908155365000} + \frac{2173589973652t_4t_8}{279897414815} + \frac{302135911297903t_6t_7}{40984127845890},
\end{aligned}$$

$$K_8^1 = \frac{2828405696582354177109987369765353t_8^{13}}{2192552418323772900665150972160000000} + \frac{104659508314126104547782052831201t_7t_8^9}{540637260904413472731230512000000} + \frac{102899753677493534416025877059t_6t_8^7}{84386669235860583672928800000} + \frac{8481094570209522684734123t_5t_8^6}{18773452555252632630240000} + \frac{6651723858574138184203661671t_7^2t_8^5}{1361075310255815865692400000} + \frac{1289206380441371567t_4t_8^4}{360580492313672000} + \frac{670532779771t_3t_8^3}{111191928400} + \frac{155409080204960966846669t_6t_7t_8^3}{17703893213525180355000} + \frac{48735507458090179297t_5t_7t_8^2}{2625716457326686000} + \frac{47752579078487t_4t_7}{5597948296300} + \frac{2427437707690958t_6^2t_8}{308166864192505} + \frac{1029963t_2t_8}{203732} + \frac{3178482572407t_5t_6}{719019786770} + \frac{253103049412822813259t_7^3t_8}{38349279837271335000},$$

$$K_3^2 = \frac{6307t_8}{2666}, K_4^2 = \frac{4500609957t_8^2}{1253739820}, K_5^2 = \frac{382063294638363t_8^4}{298801303820960} + \frac{1389601541t_7}{282855935},$$

$$K_6^2 = \frac{249245177327411089t_8^5}{818970646753180000} + \frac{281315094790593t_7t_8}{35950989338500},$$

$$K_7^2 = \frac{1688680021615354182905594447t_8^7}{8166451861534895194154400000} + \frac{367873390741219178203t_7t_8^3}{59078620289850435000} + \frac{227080200740691t_6t_8}{27322751897260} + \frac{6011337539t_5}{1131423740},$$

$$K_8^2 = \frac{513287066338916479781267251t_8^{10}}{24864125687376253900656000000} + \frac{195277479174731922608232983t_7t_8^6}{107453313967564410449400000} + \frac{766053117047317547t_6t_8^4}{296734015730570800} + \frac{2118272066471217t_5t_8^3}{298801303820960} + \frac{11494829512942367213t_7^2t_8^2}{1036467022628955000} + \frac{81503559t_4t_8}{8471215} + \frac{21253t_3}{6665} + \frac{123284247247959t_6t_7}{13661375948630},$$

$$K_4^3 = \frac{98}{37}, K_5^3 = \frac{17937969t_8^2}{8818136}, K_6^3 = \frac{5442056846681t_8^3}{1981876066000},$$

$$K_7^3 = \frac{137374164909966859t_8^5}{361870750890940000} + \frac{683860335753t_7t_8}{77489370100},$$

$$K_8^3 = \frac{7182452159729390399233t_8^8}{40426843274219975775000} + \frac{1428397083604124417t_7t_8^4}{542806126336410000} + \frac{388911287667t_6t_8^2}{39637521320} + \frac{247282497t_5t_8}{44090680} + \frac{58064641684t_7^2}{11623405515},$$

$$K_5^4 = \frac{16497t_8}{3844}, K_6^4 = \frac{1908504231t_8^2}{863939000}, K_7^4 = \frac{1510115085289879t_8^4}{946479732060000} + \frac{305059657t_7}{50668725},$$

$$K_8^4 = \frac{75610072721381020687t_8^7}{1127863107911978400000} + \frac{3665966665482509t_7t_8^3}{473239866030000} + \frac{99038823t_6t_8}{17278780} + \frac{96867t_5}{19220},$$

$$K_6^5 = \frac{86}{25}, K_7^5 = \frac{48222393t_8^2}{9129500}, K_8^5 = \frac{19962579297353t_8^5}{21758154760000} + \frac{108825561t_7t_8}{9129500},$$

$$K_7^6 = \frac{4633t_8}{1178}, K_8^6 = \frac{613126423t_8^4}{280750384} + \frac{12628t_7}{2945}, K_8^7 = \frac{85803t_8^2}{19220},$$

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